

Hopf-Galois and Bi-Galois Extensions

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1 Introduction

Hopf-Galois extensions were introduced by Chase and Sweedler [8] (in the commutative case) and Kreimer and Takeuchi [25] (in the case of finite dimensional Hopf algebras) by axioms directly generalizing those of a Galois extension of rings, replacing the action of a group on the algebra by the coaction of a Hopf algebra H ; the special case of an ordinary Galois extension is recovered by specializing H to be the dual of a group algebra. Hopf-Galois extensions also generalize strongly graded algebras (here H is a group algebra) and certain inseparable field extensions (here the Hopf algebra is the restricted envelope of a restricted Lie algebra, or, in more general cases, generated by higher derivations). They comprise twisted group rings $R * G$ of a group G acting on a ring R (possibly also twisted by a cocycle), and similar constructions for actions of Lie algebras. If the Hopf algebra involved is the coordinate ring of an affine group scheme, faithfully flat Hopf-Galois extensions are precisely the coordinate rings of affine torsors or principal homogeneous spaces. By analogy, Hopf-Galois extensions with Hopf algebra H the coordinate ring of a quantum group can be considered as the noncommutative analog of a principal homogeneous space, with a quantum group as its structure group. Apart from this noncommutative-geometric interpretation, and apart from their role as a unifying

1991 *Mathematics Subject Classification.* 16W30.

Key words and phrases. Hopf algebra, Hopf-Galois extension, bi-Galois extension.

language for many examples of good actions of things on rings, Hopf-Galois extensions are frequently used as a tool in the investigation of the structure of Hopf algebras themselves.

In this paper we try to collect some of the basic facts of the theory of Hopf-Galois extensions and (see below) bi-Galois extensions, offering alternative proofs in some instances, and proving new facts in very few instances.

In the first part we treat Hopf-Galois extensions and discuss various properties by which they can, to some extent, be characterized. After providing the necessary definitions, we first treat the special case of cleft extensions, repeating (with some more details) a rather short proof from [38] of their characterization, due to Blattner, Cohen, Doi, Montgomery, and Takeuchi [15, 5, 6]. Cleft extensions are the same as crossed products, which means that they have a combinatorial description that specializes in the case of cocommutative Hopf algebras to a cohomological description in terms of Sweedler cohomology [46].

In Section 2.3 we prove Schneider's structure theorem for Hopf modules, which characterizes faithfully flat Hopf-Galois extensions as those comodule algebras A that give rise to an equivalence of the category of Hopf modules \mathcal{M}_A^H with the category of modules of the ring of coinvariants under the coaction of H . The structure theorem is one of the most ubiquitous applications of Hopf-Galois theory in the theory of Hopf algebras. We emphasize the role of faithfully flat descent in its proof.

A more difficult characterization of faithfully flat Hopf-Galois extensions, also due to Schneider, is treated in Section 2.4. While the definition of an H -Galois extension A of B asks for a certain canonical map $\beta: A \otimes_B A \rightarrow A \otimes H$ to be bijective, it is sufficient to require it to be surjective, provided we work over a field and A is an injective H -comodule. When we think of Hopf-Galois extensions as principal homogeneous spaces with structure quantum group, this criterion has a geometric meaning. We will give a new proof for it, which is more direct than that in [44]. The new proof has two nice side-effects: First, it is more parallel to the proof that surjectivity of the canonical map is sufficient for finite-dimensional Hopf algebras (in fact so parallel that we prove the latter fact along with Schneider's result). Secondly, it yields without further work the fact that an H -Galois extension A/B that is faithfully flat as a B -module is always projective as a B -module¹.

Section 2.5 treats (a generalized version of) a characterization of Hopf-Galois extensions due to Ulbrich: An H -Galois extension of B is (up to certain additional conditions) the same thing as a monoidal functor ${}^H\mathcal{M} \rightarrow {}_B\mathcal{M}_B$ from the monoidal category of H -comodules to the category of B -bimodules.

In Section 2.6 we deal with another characterization of Hopf-Galois extensions by monoidal functors: Given any H -comodule algebra A with coinvariants B , we can define a monoidal category ${}_A\mathcal{M}_A^H$ of Hopf bimodules (monoidal with the tensor product over A), and a *weak* monoidal functor from this to the category of B -bimodules. Again up to some technical conditions, the functor is monoidal if and only if A is an H -Galois extension of B .

In Section 2.8 we show how to characterize Hopf-Galois extensions without ever mentioning a Hopf algebra. The axioms of a torsor we give here are a simplified

¹Since the present paper was submitted, the new proof has been developed further in joint work with H.-J. Schneider, in particular to also prove some results on Q -Galois extensions for a quotient coalgebra and one sided module of H ; this type of extensions will not be considered in the present paper.

variant of axioms recently introduced by Grunspan. A crucial ingredient in the characterization is again the theory of faithfully flat descent.

The second part of the paper deals with bi-Galois objects. This means, first of all, that we restrict our attention to Galois extensions of the base ring k rather than of an arbitrary coinvariant subring. Contrary, as it were, to the theory of torsors that can do without any Hopf algebras, the theory of bi-Galois extensions exploits the fact that any Hopf-Galois object has *two* rather than only one Hopf algebra in it. More precisely, for every H -Galois extension A of k there is a uniquely determined second Hopf algebra L such that A is a left L -Galois extension of A and an L - H -bicomodule. We will give an account of the theory and several ways in which the new Hopf algebra L can be applied. Roughly speaking, this may happen whenever there is a fact or a construction that depends on the condition that the Hopf algebra H be cocommutative (which, in terms of bi-Galois theory, means $L \cong H$). If this part of the cocommutative theory does not survive if H fails to be cocommutative, then maybe L can be used to replace H . Our approach will stress a very general universal property of the Hopf algebra L in an L - H -Galois extension. Several versions of this were already used in previous papers, but the general version we present here appears to be new. The construction of L was invented in the commutative case by Van Oystaeyen and Zhang to repair the failing of the fundamental theorem of Galois theory for Hopf-Galois extensions. We will discuss an application to the computation of Galois objects over tensor products, and to the problem of reducing the Hopf algebra in a Hopf-Galois object to a quotient Hopf algebra (here, however, L arises because of a lack of commutativity rather than cocommutativity). Perhaps the most important application is that bi-Galois extensions classify monoidal category equivalences between categories of comodules over Hopf algebras.

Some conventions and background facts can be found in an appendix. Before starting, however, let us point out a general notational oddity: Whenever we refer to an element $\xi \in V \otimes W$ of the tensor product of two modules, we will take the liberty to “formally” write $\xi = v \otimes w$, even if we know that the element in question is not a simple tensor, or, worse, has to be chosen from a specific submodule that is not even generated by simple tensors. Such formal notations are of course widely accepted under the name Sweedler notation for the comultiplication $\Delta(c) = c_{(1)} \otimes c_{(2)} \in C \otimes C$ in a coalgebra C , or $\delta(v) = v_{(0)} \otimes v_{(1)}$ for a right comodule, or $\delta(v) = v_{(-1)} \otimes v_{(0)}$ for a left comodule.

For a coalgebra C and a subspace $V \subset C$ we will write $V^+ = V \cap \text{Ker}(\varepsilon)$. C^{cop} denotes the coalgebra C with coopposite comultiplication, A^{op} the algebra A with opposite multiplication. Multiplication in an algebra A will be denoted by $\nabla: A \otimes A \rightarrow A$.

2 Hopf-Galois theory

2.1 Definitions. Throughout this section, H is a k -bialgebra, flat over k . A (right) H -comodule algebra A is by definition an algebra in the monoidal category of right H -comodules, that is, a right H -comodule via $\delta: A \ni a \mapsto a_{(0)} \otimes a_{(1)}$ and an algebra, whose multiplication $\nabla: A \otimes A \rightarrow A$ is a colinear map, as well as the unit $\eta: k \rightarrow A$. These conditions mean that the unit $1_A \in A$ is a coinvariant element, $1_{(0)} \otimes 1_{(1)} = 1 \otimes 1$, and that $\delta(xy) = x_{(0)}y_{(0)} \otimes x_{(1)}y_{(1)}$ holds for all $x, y \in A$. Equivalently, A is an algebra and an H -comodule in such a way that the comodule

structure is an algebra homomorphism $\delta: A \rightarrow A \otimes H$. For any H -comodule M we let $M^{\text{co}H} := \{m \in M \mid \delta(m) = m \otimes 1\}$ denote the subset of H -coinvariants. It is straightforward to check that $A^{\text{co}H}$ is a subalgebra of A .

Definition 2.1.1 The right H -comodule algebra A is said to be an H -Galois extension of $B := A^{\text{co}H}$, if the Galois map

$$\beta: A \otimes_B A \ni x \otimes y \mapsto xy_{(0)} \otimes y_{(1)} \in A \otimes H$$

is a bijection. More precisely we should speak of a right H -Galois extension; it is clear how a left H -Galois extension should be defined.

We will use the term “(right) Galois object” as shorthand for a right H -Galois extension A of k which is a faithfully flat k -module.

The first example that comes to mind is the H -comodule algebra H itself:

Example 2.1.2 Let H be a bialgebra. Then H is an H -comodule algebra, with $H^{\text{co}H} = k$. The Galois map $\beta: H \otimes H \rightarrow H \otimes H$ is the map $T(id)$, where

$$T: \text{Hom}(H, H) \rightarrow \text{End}_{H^-}^{-H}(H \otimes H)$$

is the anti-isomorphism from Lemma 4.4.1. Thus, H is a Hopf algebra if and only if the identity on H is convolution invertible if and only if the Galois map is bijective if and only if H is an H -Galois extension of k .

The notion of a Hopf-Galois extension serves to unify various types of extensions. These are recovered as we specialize the Hopf algebra H to one of a number of special types:

Example 2.1.3 Let A/k be a Galois field extension, with (finite) Galois group G . Put $H = k^G$, the dual of the group algebra. Then A is an H -Galois extension of k . Bijectivity of the Galois map $A \otimes A \rightarrow A \otimes H$ is a consequence of the independence of characters.

The definition of a Galois extension A/k of commutative rings in [9] requires (in one of its many equivalent formulations) precisely the bijectivity of the Galois map $A \otimes A \rightarrow A \otimes k^G$, beyond of course the more obvious condition that k be the invariant subring of A under the action of a finite subgroup G of the automorphism group of A . Thus Hopf-Galois extensions of commutative rings are direct generalizations of Galois extensions of commutative rings.

Example 2.1.4 Let $A = \bigoplus_{g \in G} A_g$ be a k -algebra graded by a group G . Then A is naturally an H -comodule algebra for the group algebra kG , whose coinvariant subring is $B = A_e$, the homogeneous component whose degree is the neutral element. The Galois map $A \otimes_B A \rightarrow A \otimes H$ is surjective if and only if $A_g A_h = A_{gh}$ for all $g, h \in G$, that is, A is strongly graded [10, 52]. As we shall see in Corollary 2.4.9, this condition implies that A is an H -Galois extension of B if k is a field.

We have seen already that a bialgebra H is an H -Galois extension of k if and only if it is a Hopf algebra. The following more general observation is the main result of [34]; we give a much shorter proof that is due to Takeuchi [51].

Lemma 2.1.5 *Let H be a k -flat bialgebra, and A a right H -Galois extension of $B := A^{\text{co}H}$, which is faithfully flat as k -module. Then H is a Hopf algebra.*

Proof H is a Hopf algebra if and only if the map $\beta_H: H \otimes H \ni g \otimes h \mapsto gh_{(1)} \otimes h_{(2)}$ is a bijection. By assumption the map $\beta_A: A \otimes_B A \ni x \otimes y \mapsto xy_{(0)} \otimes y_{(1)} \in A \otimes H$ is a bijection. Now the diagram

$$\begin{array}{ccc}
A \otimes_B A \otimes_B A & \xrightarrow{A \otimes_B \beta_A} & A \otimes_B A \otimes H \\
\beta_A \otimes_B A \downarrow & & \downarrow \beta_{A \otimes H} \\
(A \otimes H) \otimes_B A & & \\
(\beta_A)_{13} \downarrow & & \\
A \otimes H \otimes H & \xrightarrow{A \otimes \beta_H} & A \otimes H \otimes H
\end{array}$$

commutes, where $(\beta_A)_{13}$ denotes the map that applies β_A to the first and third tensor factor, and leaves the middle factor untouched. Thus $A \otimes \beta_H$, and by faithful flatness of A also β_H , is a bijection. \square

The Lemma also shows that if A is an H -Galois extension and a flat k -module, then A^{op} is never an H^{op} -Galois extension, unless the antipode of H is bijective. On the other hand (see [44]):

Lemma 2.1.6 *If the Hopf algebra H has bijective antipode and A is an H -comodule algebra, then A is an H -Galois extension if and only if A^{op} is an H^{op} -Galois extension.*

Proof The canonical map $A^{\text{op}} \otimes_{B^{\text{op}}} A^{\text{op}} \rightarrow A^{\text{op}} \otimes H^{\text{op}}$ identifies with the map $\beta': A \otimes_B A \rightarrow A \otimes H$ given by $\beta'(x \otimes y) = x_{(0)}y \otimes x_{(1)}$. One checks that the diagram

$$\begin{array}{ccc}
A \otimes_B A & \xrightarrow{\beta} & A \otimes H \\
& \searrow \beta' & \downarrow \alpha \\
& & A \otimes H
\end{array}$$

commutes, where $\alpha: A \otimes H \ni a \otimes h \mapsto a_{(0)} \otimes a_{(1)}S(h)$ is bijective with $\alpha^{-1}(a \otimes h) = a_{(0)} \otimes S^{-1}(h)a_{(1)}$. \square

Lemma 2.1.7 *Let A be an H -Galois extension of B . For $h \in H$ we write $\beta^{-1}(1 \otimes h) =: h^{[1]} \otimes h^{[2]}$. For $g, h \in H$, $b \in B$ and $a \in A$ we have*

$$h^{[1]}h^{[2]}_{(0)} \otimes h^{[2]}_{(1)} = 1 \otimes h \quad (2.1.1)$$

$$h^{[1]} \otimes h^{[2]}_{(0)} \otimes h^{[2]}_{(1)} = h_{(1)}^{[1]} \otimes h_{(1)}^{[2]} \otimes h_{(2)} \quad (2.1.2)$$

$$h^{[1]}_{(0)} \otimes h^{[2]} \otimes h^{[1]}_{(1)} = h_{(2)}^{[1]} \otimes h_{(2)}^{[2]} \otimes S(h_{(1)}) \quad (2.1.3)$$

$$h^{[1]}h^{[2]} = \varepsilon(h)1_A \quad (2.1.4)$$

$$(gh)^{[1]} \otimes (gh)^{[2]} = h^{[1]}g^{[1]} \otimes g^{[2]}h^{[2]} \quad (2.1.5)$$

$$bh^{[1]} \otimes h^{[2]} = h^{[1]} \otimes h^{[2]}b \quad (2.1.6)$$

$$a_{(0)}a_{(1)}^{[1]} \otimes a_{(1)}^{[2]} = 1 \otimes a \quad (2.1.7)$$

We will omit the proof, which can be found in [45, (3.4)].

Definition 2.1.8 Let H be a Hopf algebra, and A an H -Galois extension of B . The Miyashita-Ulbrich action of H on the centralizer A^B of B in A is given by $x \leftarrow h = h^{[1]}xh^{[2]}$ for $x \in A^B$ and $h \in H$.

The expression $h^{[1]}xh^{[2]}$ is well-defined because $x \in A^B$, and it is in A^B again because $h^{[1]} \otimes h^{[2]} \in (A \otimes_B A)^B$. The following properties of the Miyashita-Ulbrich action can be found in [52, 16] in different language.

Lemma 2.1.9 *The Miyashita-Ulbrich action makes A^B an object of \mathcal{YD}_H^H , and thus the weak center of the monoidal category \mathcal{M}^H of right H -comodules. A^B is the center of A in the sense of Definition 4.2.1.*

Proof It is trivial to check that A^B is a subcomodule of A . It is a Yetter-Drinfeld module by (2.1.3) and (2.1.2). Now the inclusion $A^B \hookrightarrow A$ is central in the sense of Definition 4.2.1, since $a_{(0)}(x \leftarrow a_{(1)}) = a_{(0)}a_{(1)}^{[1]}xa_{(1)}^{[2]} = xa$ for all $a \in A$ and $x \in A^B$ by (2.1.7). Finally let us check the universal property in Definition 4.2.1: Let V be a Yetter-Drinfeld module, and $f: V \rightarrow A$ an H -colinear map with $a_{(0)}f(v \leftarrow a_{(1)}) = f(v)a$ for all $v \in V$ and $a \in A$. Then we see immediately that f takes values in A^B . Moreover, we have $f(v) \leftarrow h = h^{[1]}f(v)h^{[2]} = h^{[1]}h^{[2]}_{(0)}f(v \leftarrow h^{[2]}_{(1)}) = f(v \leftarrow h)$ for all $h \in H$ by (2.1.1). \square

Much of the ‘‘meaning’’ of the Miyashita-Ulbrich action can be guessed from the simplest example $A = H$. Here we have $h^{[1]} \otimes h^{[2]} = S(h_{(1)}) \otimes h_{(2)} \in H \otimes H$, and thus the Miyashita-Ulbrich action is simply the adjoint action of H on itself.

2.2 Cleft extensions and crossed products. Throughout the section, H is a k -bialgebra.

Definition 2.2.1 Let B be a k -algebra. A map $\rightharpoonup: H \otimes B \rightarrow B$ is a measuring if $h \rightharpoonup (bc) = (h_{(1)} \rightharpoonup b)(h_{(2)} \rightharpoonup c)$ and $h \rightharpoonup 1 = 1$ hold for all $h \in H$ and $b, c \in B$.

Let H be a bialgebra, and B an algebra. A crossed product $B \#_{\sigma} H$ is the structure of an associative algebra with unit $1 \# 1$ on the k -module $B \#_{\sigma} H := B \otimes H$, in which multiplication has the form

$$(b \# g)(c \# h) = b(g_{(1)} \rightharpoonup c)\sigma(g_{(2)} \otimes h_{(1)}) \# g_{(3)}h_{(2)}$$

for some measuring $\rightharpoonup: H \otimes B \rightarrow B$ and some linear map $\sigma: H \otimes H \rightarrow B$.

We have quite deliberately stated the definition without imposing any explicit conditions on σ . Such conditions are implicit, however, in the requirement that multiplication be associative and have the obvious unit. We have chosen the definition above to emphasize that the explicit conditions on σ are never used in our approach to the theory of crossed products. They are, however, known and not particularly hard to derive:

Proposition 2.2.2 *Let H be a bialgebra, $\rightharpoonup: H \otimes B \rightarrow B$ a measuring, and $\sigma: H \otimes H \rightarrow B$ a k -linear map. The following are equivalent:*

1. $A = B \# H := B \otimes H$ is an associative algebra with unit $1 \# 1$ and multiplication

$$(b \# g)(c \# h) = b(g_{(1)} \rightharpoonup c)\sigma(g_{(2)} \otimes h_{(1)}) \# g_{(3)}h_{(2)}.$$

2. (a) \rightharpoonup is a twisted action, that is $(g_{(1)} \rightharpoonup (h_{(1)} \rightharpoonup b))\sigma(g_{(2)} \otimes h_{(2)}) = \sigma(g_{(1)} \otimes h_{(1)})(g_{(2)}h_{(2)} \rightharpoonup b)$ and $1 \rightharpoonup b = b$ hold for all $g, h \in H$ and $b \in B$.

- (b) σ is a two-cocycle, that is $(f_{(1)} \rightharpoonup \sigma(g_{(1)} \otimes h_{(1)}))\sigma(f_{(2)} \otimes g_{(2)}h_{(2)}) = \sigma(f_{(1)} \otimes g_{(1)})\sigma(f_{(2)} \otimes g_{(2)} \otimes h)$ and $\sigma(h \otimes 1) = \sigma(1 \otimes h) = 1$ hold for all $f, g, h \in H$.

Not only are the conditions on σ known, but, more importantly, they have a cohomological interpretation in the case where H is cocommutative and B is commutative. In this case a twisted action is clearly simply a module algebra structure. Sweedler [46] has defined cohomology groups $H^\bullet(H, B)$ for a cocommutative bialgebra H and commutative H -module algebra B , and it turns out that a convolution invertible map σ as above is precisely a two-cocycle in this cohomology. Sweedler's paper also contains the construction of a crossed product from a two-cocycle, and the fact that his second cohomology group classifies cleft extensions (which we shall define below) by assigning the crossed product to a cocycle. Group cohomology with coefficients in the unit group of B as well as (under some additional conditions) Lie algebra cohomology with coefficients in the additive group of B are examples of Sweedler cohomology, and the cross product construction also has precursors for groups (twisted group rings with cocycles, which feature in the construction of elements of the Brauer group from group cocycles) and Lie algebras. Thus, the crossed product construction from cocycles can be viewed as a nice machinery producing (as we shall see shortly) Hopf-Galois extensions *in the case of cocommutative Hopf algebras and commutative coinvariant subrings*. In the general case, the equations do not seem to have any reasonable cohomological interpretation, so while cleft extensions remain an important special class of Hopf-Galois extensions, it is rarely possible to construct them by finding cocycles in some conceptually pleasing way.

We now proceed to prove the characterization of crossed products as special types of comodule algebras, which is due to Blattner, Cohen, Doi, Montgomery, and Takeuchi:

Definition 2.2.3 Let A be a right H -comodule algebra, and $B := A^{\text{co}H}$.

1. A is cleft if there exists a convolution invertible H -colinear map $j: H \rightarrow A$ (also called a cleaving).
2. A normal basis for A is an H -colinear and B -linear isomorphism $\psi: B \otimes H \rightarrow A$.

If \tilde{j} is a cleaving, then $\tilde{j}(1)$ is a unit in B , and thus $j(h) = \tilde{j}(1)^{-1}\tilde{j}(h)$ defines another cleaving, which, moreover, satisfies $j(1) = 1$.

It was proved by Doi and Takeuchi [15] that A is H -Galois with a normal basis if and only if it is cleft, and in this case A is a crossed product $A \cong B \#_\sigma H$ with an invertible cocycle $\sigma: H \otimes H \rightarrow B$. Blattner and Montgomery [6] have shown that crossed products with an invertible cocycle are cleft.

Clearly a crossed product is always an H -comodule algebra with an obvious normal basis.

Lemma 2.2.4 Assume that the H -comodule algebra A has a normal basis $\psi: B \otimes H \rightarrow A$ satisfying $\psi(1 \otimes 1) = 1$. Then A is isomorphic (via ψ) to a crossed product.

Proof In fact we may as well assume $B \otimes H = A$ as B -modules and H -comodules. Define $h \rightharpoonup b = (B \otimes \varepsilon)((1 \otimes h)(b \otimes 1))$ and $\sigma(g \otimes h) = (B \otimes \varepsilon)((1 \otimes$

$g)(1 \otimes h)$). Since multiplication is H -colinear, we find

$$\begin{aligned} (1 \otimes g)(c \otimes 1) &= (B \otimes \varepsilon \otimes H)(B \otimes \Delta)((1 \otimes g)(c \otimes 1)) \\ &= (B \otimes \varepsilon \otimes H)((1 \otimes g_{(1)})(b \otimes 1) \otimes g_{(2)}) = g_{(1)} \rightharpoonup b \otimes g_{(2)}, \end{aligned}$$

$$\begin{aligned} (1 \otimes g)(1 \otimes h) &= (B \otimes \varepsilon \otimes H)(B \otimes \Delta)((1 \otimes g)(1 \otimes h)) \\ &= (B \otimes \varepsilon \otimes H)((1 \otimes g_{(1)})(1 \otimes h_{(1)}) \otimes g_{(2)}h_{(2)}) = \sigma(g_{(1)} \otimes h_{(1)}) \otimes g_{(2)}h_{(2)}, \end{aligned}$$

and finally

$$\begin{aligned} (b \otimes g)(c \otimes h) &= (b \otimes 1)(1 \otimes g)(c \otimes 1)(1 \otimes h) = (b \otimes 1)(g_{(1)} \rightharpoonup c \otimes g_{(2)})(1 \otimes h) \\ &= b(g_{(1)} \rightharpoonup c)\sigma(g_{(2)} \otimes h_{(1)}) \otimes g_{(3)}h_{(2)}. \end{aligned}$$

□

To prove the remaining parts of the characterization, we will make heavy use of the isomorphisms T_A^C from Lemma 4.4.1, for various choices of algebras A and coalgebras C .

Lemma 2.2.5 *Let $j: H \rightarrow A$ be a cleaving. Then there is a normal basis $\psi: B \otimes H \rightarrow A$ with $j = \psi(\eta_B \otimes H)$. If $j(1) = 1$, then $\psi(1 \otimes 1) = 1$.*

Proof We claim that $\psi: B \otimes H \ni b \otimes h \mapsto bj(h) \in A$ is a normal basis.

Since the comodule structure $\delta: A \rightarrow A \otimes H$ is an algebra map, δj is convolution invertible. Moreover $\delta j = (j \otimes H)\Delta$ by assumption. For $a \in A$, we have

$$\begin{aligned} T_{A \otimes H}^H(\delta j)(\delta(a_{(0)}j^{-1}(a_{(1)})) \otimes a_{(2)}) &= T_{A \otimes H}^H(\delta j)T_{A \otimes H}^H(\delta j^{-1})(\delta(a_{(0)}) \otimes a_{(1)}) \\ &= a_{(0)}j^{-1}(a_{(1)})j(a_{(2)}) \otimes a_{(3)} \otimes a_{(4)} = T_{A \otimes H}^H((j \otimes H)\Delta)(a_{(0)}j^{-1}(a_{(1)}) \otimes 1 \otimes a_{(2)}), \end{aligned}$$

hence $\delta(a_{(0)}j^{-1}(a_{(1)})) \otimes a_{(2)} = a_{(0)}j^{-1}(a_{(1)}) \otimes a_{(2)}$, and further $\delta(a_{(0)}j^{-1}(a_{(1)})) = a_{(0)}j^{-1}(a_{(1)}) \otimes 1$. Thus $A \ni a \mapsto a_{(0)}j^{-1}(a_{(1)}) \otimes a_{(2)} \in B \otimes H$ is well defined and easily checked to be an inverse for ψ . □

Lemma 2.2.6 *Let A be an H -comodule algebra with a normal basis. Put $B := A^{\text{co}H}$. The following are equivalent:*

1. A is an H -Galois extension of B .
2. A is cleft.

Proof We can assume that $A = B \#_{\sigma} H$ is a crossed product, and that $j(h) = 1 \otimes h$.

The map $\alpha: B \otimes H \otimes H \rightarrow A \otimes_B A$ with $\alpha(b \otimes g \otimes h) = b \otimes g \otimes 1 \otimes h$ is an isomorphism. For $b \in B$ and $g, h \in H$ we have

$$\beta_A \alpha(b \otimes g \otimes h) = \beta_A(b \otimes g \otimes j(h)) = (b \otimes g)j(h_{(1)}) \otimes h_{(2)} = T_A^H(j)(b \otimes g \otimes h),$$

that is $\beta_A \alpha = T_A^H(j)$. In particular, β_A is an isomorphism if and only if $T_A^H(j)$ is, if and only if j is convolution invertible. □

In particular, if B is faithfully flat over k , then cleft extensions can only occur if H is a Hopf algebra. If this is the case, we find:

Theorem 2.2.7 *Let H be a Hopf algebra and A a right H -comodule algebra with $B := A^{\text{co}H}$. The following are equivalent:*

1. A is H -cleft.
2. A is H -Galois with a normal basis.

3. A is isomorphic to a crossed product $B\#_\sigma H$ such that the cocycle $\sigma: H \otimes H \rightarrow B$ is convolution invertible.

Proof We have already shown that under any of the three hypotheses we can assume that $A \cong B\#_\sigma H = B \otimes H$ is a crossed product, with $j(h) = 1 \otimes h$, and we have seen that (1) is equivalent to (2), even if H does not have an antipode.

Now, for $b \in B, g, h \in H$ we calculate

$$\begin{aligned} T_A^H(j)(b \otimes g \otimes h) &= bj(g)j(h_{(1)}) \otimes h_{(2)} = b\sigma(g_{(1)} \otimes h_{(1)}) \otimes g_{(2)}h_{(2)} \otimes h_{(3)} \\ &= (B \otimes \beta_H)(b\sigma(g_{(1)} \otimes h_{(1)}) \otimes g_{(2)} \otimes h_{(2)}) = (B \otimes \beta_H)T_B^{H \otimes H}(\sigma)(b \otimes g \otimes h) \end{aligned}$$

that is, $T_A^H(j) = (B \otimes \beta_H)T_B^{H \otimes H}(\sigma)$. Since we assume that β_H is a bijection, we see that j is convolution invertible if and only if $T_A^H(j)$ is bijective, if and only if $T_B^{H \otimes H}(\sigma)$ is bijective, if and only if σ is convolution invertible. \square

The reader that has seen the proof of (3) \Rightarrow (1) in [6] may be worried that we have lost some information: In [6] the convolution inverse of j is given explicitly, while we only seem to have a rather roundabout existence proof. However, we see from our arguments above that

$$j^{-1} = (T_A^H)^{-1} (T_B^{H \otimes H}(\sigma^{-1})(B \otimes \beta_H^{-1})),$$

that is,

$$\begin{aligned} j^{-1}(h) &= (A \otimes \varepsilon)T_B^{H \otimes H}(\sigma^{-1})(B \otimes \beta_H^{-1})(1 \otimes 1 \otimes h) \\ &= (A \otimes \varepsilon)T_B^{H \otimes H}(\sigma^{-1})(1 \otimes S(h_{(1)}) \otimes h_{(2)}) \\ &= (A \otimes \varepsilon)(\sigma^{-1}(S(h_{(2)}) \otimes h_{(3)}) \otimes S(h_{(1)}) \otimes h_{(4)}) \\ &= \sigma^{-1}(S(h_{(2)}) \otimes h_{(3)})\#S(h_{(1)}). \end{aligned}$$

2.3 Descent and the structure of Hopf modules.

Definition 2.3.1 Let A be a right H -comodule algebra. A Hopf module $M \in \mathcal{M}_A^H$ is a right A -module in the monoidal category of H -comodules. That is, M is a right H -comodule and a right A -module such that the module structure is an H -colinear map $M \otimes A \rightarrow M$. This in turn means that $\delta(ma) = m_{(0)}a_{(0)} \otimes m_{(1)}a_{(1)}$ holds for all $m \in M$ and $a \in A$.

For any comodule algebra, one obtains a pair of adjoint functors between the category of Hopf modules and the category of modules over the coinvariant subalgebra.

Lemma 2.3.2 *Let H be a k -flat Hopf algebra, A a right H -comodule algebra, and $B = A^{\text{co}H}$. Then the functor*

$$\mathcal{M}_A^H \ni M \mapsto M^{\text{co}H} \in \mathcal{M}_B$$

is right adjoint to

$$\mathcal{M}_B \ni N \mapsto N \otimes_B A \in \mathcal{M}_A^H$$

Here, both the A -module and H -comodule structures of $N \otimes_B A$ are induced by those of A . The unit and counit of the adjunction are

$$N \ni n \mapsto n \otimes 1 \in (N \otimes_B A)^{\text{co}H}$$

$$M^{\text{co}H} \otimes_B A \ni m \otimes a \mapsto ma \in M$$

If the adjunction in the Lemma is an equivalence, then we shall sometimes say that the structure theorem for Hopf modules holds for the extension. A theorem of Schneider [44] characterizes faithfully flat Hopf-Galois extensions as those comodule algebras for which the adjunction above is an equivalence. The proof in [44] uses faithfully flat descent; we rewrite it to make direct use of the formalism of faithfully flat descent of modules that we recall in Section 4.5. This approach was perhaps first noted in my thesis [32], though it is certainly no surprise; in fact, one of the more prominent special cases of the structure theorem for Hopf modules over Hopf-Galois extensions that is one direction of the characterization goes under the name of Galois descent.

Example 2.3.3 Let A/k be a Galois field extension with Galois group G . A comodule structure making an A -vector space into a Hopf module $M \in \mathcal{M}_A^{kG}$ is the same as an action of the Galois group G on M by semilinear automorphisms, i.e. in such a way that $\sigma \cdot (am) = \sigma(a)(\sigma \cdot m)$ holds for all $m \in M$, $a \in A$ and $\sigma \in G$. Galois descent (see for example [23]) says, most of all, that such an action on M forces M to be obtained from a k -vector space by extending scalars. This is (part of) the content of the structure theorem for Hopf modules.

Remark 2.3.4 Let A be an H -comodule algebra; put $B := A^{\text{co}H}$. As a direct generalization of the Galois map $\beta: A \otimes_B A \rightarrow A \otimes H$, we have a right A -module map

$$\beta_M: M \otimes_B A \ni m \otimes a \mapsto ma_{(0)} \otimes a_{(1)} \in M \otimes H.$$

which is natural in $M \in \mathcal{M}_A$. Of course, the Galois map is recovered as $\beta = \beta_A$. Note that β_M can be identified with $M \otimes_A \beta_A$, so that all β_M are bijective once β_A is bijective.

Lemma 2.3.5 *Let A be a right H -comodule algebra, and $B := A^{\text{co}H}$. For each descent data $(M, \theta) \in \mathcal{D}(A \downarrow B)$, the map*

$$\delta := \left(M \xrightarrow{\theta} M \otimes_B A \xrightarrow{\beta_M} M \otimes H \right)$$

is a right H -comodule structure on M making $M \in \mathcal{M}_A^H$.

Thus, we have defined a functor $\mathcal{D}(A \downarrow B) \rightarrow \mathcal{M}_A^H$.

If A is an H -Galois extension of B , then the functor is an equivalence.

Proof Let $\theta: M \rightarrow M \otimes_B A$ be a right A -module map, and $\delta := \beta_M \theta$. Of course δ is a right A -module map, so that M is a Hopf module if and only if it is a comodule.

Now we have the commutative diagrams

$$\begin{array}{ccccc} M & \xrightarrow{\theta} & M \otimes_B A & \xrightarrow{\theta \otimes_B A} & M \otimes_B A \otimes_B A \\ & \searrow \delta & \downarrow \beta_M & \searrow \delta \otimes_B A & \downarrow \beta_{M \otimes_B A} \\ & & M \otimes H & & (M \otimes H) \otimes_B A \\ & & & \searrow \delta \otimes H & \downarrow \beta_{M \otimes H} \\ & & & & M \otimes H \otimes H \end{array}$$

using naturality of β with respect to the right A -module map δ , and

$$\begin{array}{ccccc}
 M & \xrightarrow{\theta} & M \otimes_B A & \xrightarrow{\eta_2} & M \otimes_B A \otimes_B A \\
 & \searrow \delta & \downarrow \beta_M & \searrow M \otimes \eta \otimes_B A & \downarrow \beta_{M \otimes_B A} \\
 & & M \otimes H & & (M \otimes H) \otimes_B A \\
 & & & \searrow M \otimes \Delta & \downarrow \beta_{M \otimes H} \\
 & & & & M \otimes H \otimes H
 \end{array}$$

using $\beta_{M \otimes H}(m \otimes 1 \otimes a) = (m \otimes 1)a_{(0)} \otimes a_{(1)} = ma_{(0)} \otimes a_{(1)} \otimes a_{(2)} = (M \otimes \Delta)\beta_M(m \otimes a)$. Moreover

$$\begin{array}{ccccc}
 M & \xrightarrow{\theta} & M \otimes_B A & \xrightarrow{\mu} & M \\
 & \searrow \delta & \downarrow \beta_M & \nearrow M \otimes \varepsilon & \\
 & & M \otimes H & &
 \end{array}$$

also commutes. Thus, if θ is a descent data, then δ is a comodule.

Conversely, if β is bijective, then the natural transformation β_M is an isomorphism. In particular the formula $\delta = \beta_M \theta$ defines a bijective correspondence between A -module maps $\theta: M \rightarrow M \otimes_B A$ and $\delta: M \rightarrow M \otimes H$. The same diagrams as above show that δ is a comodule structure if and only if θ is a descent data. \square

Schneider's structure theorem for Hopf modules is now an immediate consequence of faithfully flat descent:

Corollary 2.3.6 *The following are equivalent for an H -comodule algebra A :*

1. A is an H -Galois extension of $B := A^{\text{co}H}$, and faithfully flat as left B -module.
2. The functor $\mathcal{M}_B \ni N \mapsto N \otimes_B A \in \mathcal{M}_A^H$ is an equivalence.

Proof (1) \Rightarrow (2): We have established an equivalence $\mathcal{D}(A \downarrow B) \rightarrow \mathcal{M}_A^H$, and it is easy to check that the diagram

$$\begin{array}{ccc}
 \mathcal{D}(A \downarrow B) & \xrightarrow{\sim} & \mathcal{M}_A^H \\
 & \searrow (-)^\theta & \swarrow (-)^{\text{co}H} \\
 & & \mathcal{M}_B
 \end{array} \tag{2.3.1}$$

commutes. Thus the coinvariants functor is an equivalence by faithfully flat descent.

(2) \Rightarrow (1): Since $(-)^{\text{co}H}: \mathcal{M}_A^H \rightarrow \mathcal{M}_B$ is an equivalence, and $\beta: A \otimes_B A \rightarrow A \otimes H$ is a Hopf module homomorphism, β is an isomorphism if and only if $\beta^{\text{co}H}$ is. But

$$A \cong A \otimes_B A^{\text{co}H} \xrightarrow{\beta^{\text{co}H}} A \otimes H^{\text{co}H} \cong A$$

is easily checked to be the identity. Thus, A is an H -Galois extension of B . It is faithfully flat since $(-) \otimes_B A: \mathcal{M}_B \rightarrow \mathcal{M}_A^H$ is an equivalence. \square

To shed some further light on the connection between descent data and the Galois map, it may be interesting to prove a partial converse to Lemma 2.3.5:

Proposition 2.3.7 *Let H be a bialgebra, A a right H -comodule algebra, and $B = A^{\text{co}H}$.*

If the natural functor $\mathcal{D}(A \downarrow B) \rightarrow \mathcal{M}_A^H$ is an equivalence, then the Galois map $\beta: A \otimes_B A \rightarrow A \otimes H$ is surjective.

If, moreover, A is flat as left B -module, then A is an H -Galois extension of B .

Proof By assumption there is an A -module map $\theta = \theta_M: M \rightarrow M \otimes_B A$, natural in $M \in \mathcal{M}_A^H$, such that the H -comodule structure of M is given by $\delta_M = \beta_M \theta_M$.

Specializing $M = V \bullet \otimes A$ for $V \in \mathcal{M}^H$, we obtain a natural A -module map $\tilde{\theta}_V: V \otimes A \rightarrow V \otimes A \otimes_B A$, which, being an A -module map, is determined by $\phi_V: V \rightarrow V \otimes A \otimes_B A$. Finally, since ϕ_V is natural, it has the form

$$\phi_V(v) = v_{(0)} \otimes v_{(1)}^{[1]} \otimes v_{(1)}^{[2]}$$

for the map $\gamma: H \ni h \mapsto h^{[1]} \otimes h^{[2]} \in A \otimes_B A$ defined by $\gamma = (\varepsilon \otimes A \otimes_B A) \phi_H$. In particular we have $\tilde{\theta}_V(v \otimes a) = v_{(0)} \otimes v_{(1)}^{[1]} \otimes v_{(1)}^{[2]} a$, and hence, specializing $V = H$ and $a = 1$:

$$\begin{aligned} h_{(1)} \otimes 1 \otimes h_{(2)} &= \delta_{H \otimes A}(h \otimes 1) \\ &= \beta_{H \otimes A} \theta_{H \otimes A}(h \otimes 1) \\ &= \beta_{H \otimes A}(h_{(1)} \otimes h_{(2)}^{[1]} \otimes h_{(2)}^{[2]}) \\ &= h_{(1)} \otimes h_{(2)}^{[1]} h_{(2)}^{[2]}{}_{(0)} \otimes h_{(2)}^{[2]}{}_{(1)} \\ &= h_{(1)} \otimes \beta_A(h^{[1]} \otimes h^{[2]}) \end{aligned}$$

for all $h \in H$, and thus $\beta(ah^{[1]} \otimes h^{[2]}) = a \otimes h$ for all $a \in A$.

If A is left B -flat, then $\theta_M(m) \in M^\theta \otimes_B A \subset M^{\text{co}H} \otimes_B A$ implies, in particular, that $a_{(0)} \otimes a_{(1)}^{[1]} \otimes a_{(1)}^{[2]} \in (A \otimes A)^{\text{co}H} \otimes_B A$, and thus $a_{(0)} a_{(1)}^{[1]} \otimes a_{(1)}^{[2]} \in B \otimes A$. Hence $\beta^{-1}(a \otimes h) = ah^{[1]} \otimes h^{[2]}$ is actually (not only right) inverse to β by the calculation $\beta^{-1}\beta(x \otimes y) = \beta^{-1}(xy_{(0)} \otimes y_{(1)}) = xy_{(0)} y_{(1)}^{[1]} \otimes y_{(1)}^{[2]} = x \otimes y_{(0)} y_{(1)}^{[1]} y_{(1)}^{[2]} = x \otimes y$. \square

2.4 Coflat Galois extensions. A faithfully flat H -Galois extension is easily seen to be a faithfully coflat H -comodule:

Lemma 2.4.1 *Let H be a k -flat Hopf algebra, and A an H -Galois extension of B . If A_B is faithfully flat and A is a faithfully flat k -module, then A is a faithfully coflat H -comodule.*

Proof If A_B is flat, then we have an isomorphism, natural in $V \in {}^H\mathcal{M}$:

$$A \otimes_B (A \square_H V) \cong (A \otimes_B A) \square_H V \cong (A \otimes H) \square_H V \cong A \otimes V.$$

If A_B is faithfully flat and A is faithfully flat over k , then it follows that the functor $A \square_H -$ is exact and reflects exact sequences. \square

The converse is trivial if $B = k$, for then any (faithfully) coflat comodule is a (faithfully) flat k -module by the definition we chose for coflatness. This is not at all clear if B is arbitrary. However, it is true if k is a field. In this case much more can be said. Schneider [44] has proved that a coflat H -comodule algebra A is already a faithfully flat (on either side) Hopf-Galois extension if we only assume that the Galois map is surjective, and the antipode of H is bijective. We will give

a different proof of this characterization of faithfully flat Hopf-Galois extensions. Like the original, it is based on Takeuchi's result that coflatness and injectivity coincide for comodules if k is a field, and on a result of Doi on injective comodule algebras (for which, again, we will give a slightly different proof). Our proof of Schneider's criterion will have a nice byproduct: In the case that k is a field and the Hopf algebra H has bijective antipode, every faithfully flat H -Galois extension is a projective module (on either side) over its coinvariants.

Before going into any details, let us comment very briefly on the algebro-geometric meaning of Hopf-Galois extensions and the criterion. If H is the (commutative) Hopf algebra representing an affine group scheme G , A the algebra of an affine scheme X on which H acts, and Y the affine scheme represented by $A^{\text{co}H}$, then A is a faithfully flat H -Galois extension of B if and only if the morphism $X \rightarrow Y$ is faithfully flat, and the map $X \times G \rightarrow X \times_Y X$ given on elements by $(x, g) \mapsto (x, xg)$ is an isomorphism of affine schemes. This means that X is an affine scheme with an action of G and a projection to the invariant quotient Y which is locally trivial in the faithfully flat topology (becomes trivial after a faithfully flat extension of the base Y). This is the algebro-geometric version of a principal fiber bundle with structure group G , or a G -torsor [11]. If we merely require the canonical map $A \otimes A \rightarrow A \otimes H$ to be surjective, this means that we require the map $X \times G \rightarrow X \times X$ given by $(x, g) \mapsto (x, xg)$ to be a closed embedding, or that we require the action of G on X to be free. Thus, the criterion we are dealing with in this section says that under the coflatness condition on the comodule structure freeness of the action is sufficient to have a principal fiber bundle. Note in particular that surjectivity of the canonical map is trivial in the case where H is a quotient Hopf algebra of a Hopf algebra A (or G is a closed subgroup scheme of an affine group scheme X), while coflatness in this case is a representation theoretic condition (the induction functor is exact). See [44] for further literature.

For the rest of this section, we assume that k is a field.

We start by an easy and well-known observation regarding projectivity of modules over a Hopf algebra.

Lemma 2.4.2 *Let H be a Hopf algebra and $M, P \in {}_H\mathcal{M}$ with P projective. Then $\cdot P \otimes \cdot M \in {}_H\mathcal{M}$ is projective. In particular, H is semisimple if and only if the trivial H -module is projective.*

Proof The second statement follows from the first, since every module is its own tensor product with the trivial module. The diagonal module $\cdot H \otimes \cdot M$ is free by the structure theorem for Hopf modules, or since

$$\cdot H \otimes M \ni h \otimes m \mapsto h_{(1)} \otimes h_{(2)} m \in \cdot H \otimes \cdot M$$

is an isomorphism. Since any projective P is a direct summand of a direct sum of copies of H , the general statement follows. \square

If H has bijective antipode, then in the situation of the Lemma also $M \otimes P$ is projective.

For our proof, we will need the dual variant. To prepare, we observe:

Lemma 2.4.3 *Let C be a coalgebra and $M \in \mathcal{M}^C$. Then M is injective if and only if it is a direct summand of $V \otimes C^* \in \mathcal{M}^C$ for some $V \in \mathcal{M}_k$. In particular, if M is injective, then so is every $V \otimes M^* \in \mathcal{M}^C$ for $V \in \mathcal{M}_k$.*

Lemma 2.4.4 *Let H be a Hopf algebra and $M, I \in \mathcal{M}^H$ with I injective. Then $M^\bullet \otimes I^\bullet \in \mathcal{M}^H$ is injective.*

Proof Since I is a direct summand of some $V \otimes H^\bullet$, it is enough to treat the case $I = H$. But then

$$M^\bullet \otimes H^\bullet \ni m \otimes h \mapsto m_{(0)} \otimes m_{(1)} h \in M \otimes H^\bullet$$

is a colinear bijection, and $M \otimes H^\bullet$ is injective. \square

We come to a key property of comodule algebras that are injective comodules, which is due to Doi [13]:

Proposition 2.4.5 *Let H be a Hopf algebra and A an H -comodule algebra that is an injective H -comodule. Then every Hopf module in \mathcal{M}_A^H is an injective H -comodule. If H has bijective antipode, then also every Hopf module in ${}_A\mathcal{M}^H$ is an injective H -module.*

Proof Let $M \in \mathcal{M}_A^H$. Since the module structure $\mu: M^\bullet \otimes A^\bullet \rightarrow M$ is H -colinear, and splits as a colinear map via $M \ni m \mapsto m \otimes 1 \in M \otimes A$, the comodule M is a direct summand of the diagonal comodule $M \otimes A$. The latter is injective, since A is. The statement on Hopf modules in ${}_A\mathcal{M}^H$ follows since H^{op} is a Hopf algebra and we can identify ${}_A\mathcal{M}^H$ with $\mathcal{M}_{A^{\text{op}}}^{H^{\text{op}}}$. \square

Lemma 2.4.6 *The canonical map $\beta_0: A \otimes A \rightarrow A \otimes H$ is a morphism of Hopf modules in ${}_A\mathcal{M}^H$ if we equip its source and target with the obvious left A -module structures, the source with the comodule structure coming from the left tensor factor, and its target with the comodule structure given by $(a \otimes h)_{(0)} \otimes (a \otimes h)_{(1)} = a_{(0)} \otimes h_{(2)} \otimes a_{(1)} S(h_{(1)})$. The latter can be viewed as a codiagonal comodule structure, if we first endow H with the comodule structure restricted along the antipode. Thus we may write briefly that*

$$\beta_0: .A^\bullet \otimes A \rightarrow .A^\bullet \otimes H^S$$

is a morphism in ${}_A\mathcal{M}^H$.

Proposition 2.4.7 *Let H be a Hopf algebra, and A a right H -comodule algebra; put $B := A^{\text{co}H}$. Assume there is an H -comodule map $\gamma: H^S \rightarrow A^\bullet \otimes A$ such that $\beta(\gamma(h)) = 1 \otimes h$ for all $h \in H$ (where we abuse notations and also consider $\gamma(h) \in A \otimes_B A$).*

Then the counit $M^{\text{co}H} \otimes_B A \rightarrow M$ of the adjunction in Lemma 2.3.2 is an isomorphism for every $M \in \mathcal{M}_A^H$. Its inverse lifts to a natural transformation $M \rightarrow M^{\text{co}H} \otimes A$ (with the tensor product over k).

In particular A is an H -Galois extension of B , and a projective left B -module.

Proof We shall write $\gamma(h) =: h^{[1]} \otimes h^{[2]}$. This is to some extent an abuse of notations, since the same symbol was used for the map $H \rightarrow A \otimes_B A$ induced by the inverse of the canonical map in a Hopf-Galois extension. However, the abuse is not so bad, because in fact the map we use in the present proof will turn out to induce that inverse. Our assumptions on γ read $h_{(2)}^{[1]} \otimes h_{(2)}^{[2]} \otimes S(h_{(1)}) = h^{[1]}_{(0)} \otimes h^{[2]} \otimes h^{[1]}_{(1)}$ and $h^{[1]} h^{[2]}_{(0)} \otimes h^{[2]}_{(1)} = 1 \otimes h \in A \otimes H$ for all $h \in H$. The latter implies in particular that $h^{[1]} h^{[2]} = \varepsilon(h) 1_A$.

It follows for all $m \in M \in \mathcal{M}_A^H$ that $m_{(0)}m_{(1)}^{[1]} \otimes m_{(1)}^{[2]} \in M^{\text{co}H} \otimes A$; indeed $\rho(m_{(0)}m_{(1)}^{[1]} \otimes m_{(1)}^{[2]}) = m_{(0)}m_{(3)}^{[1]} \otimes m_{(1)}S(m_{(2)}) \otimes m_{(3)}^{[2]} = m_{(0)}m_{(1)}^{[1]} \otimes 1 \otimes m_{(1)}^{[2]}$.

Now we can write down the natural transformation $\psi: M \ni m \mapsto m_{(0)}m_{(1)}^{[1]} \otimes m_{(1)}^{[2]} \in M^{\text{co}H} \otimes A$, and define $\vartheta: M \rightarrow M^{\text{co}H} \otimes_B A$ as the composition of ψ with the canonical surjection.

We claim that ϑ is inverse to the adjunction map $\phi: M^{\text{co}H} \otimes_B A \rightarrow M$.

Indeed

$$\phi\vartheta(m) = \phi(m_{(0)}m_{(1)}^{[1]} \otimes m_{(1)}^{[2]}) = m_{(0)}m_{(1)}^{[1]}m_{(1)}^{[2]} = m$$

and

$$\vartheta\phi(n \otimes a) = \phi^{-1}(na) = na_{(0)}a_{(1)}^{[1]} \otimes a_{(1)}^{[2]} = n \otimes a_{(0)}a_{(1)}^{[1]}a_{(1)}^{[2]} = n \otimes a,$$

using that $a_{(0)}a_{(1)}^{[1]} \otimes a_{(1)}^{[2]} \in B \otimes A$.

Since the adjunction map is an isomorphism, A is an H -Galois extension of B .

The instance $\psi_A: A \ni a \mapsto a_{(0)}a_{(1)}^{[1]} \otimes a_{(2)}^{[2]} \in B \otimes A$ of ψ splits the multiplication map $B \otimes A \rightarrow A$, so that A is a direct summand of $B \otimes A$ as left B -module, and hence a projective B -module. \square

Corollary 2.4.8 *Let H be a Hopf algebra and A a right H -comodule algebra such that the canonical map $\beta_0: A \otimes A \rightarrow A \otimes H$ is a surjection.*

Assume in addition that $\beta_0: A^ \otimes A \rightarrow A^* \otimes H^S$ splits as a comodule map for the indicated H -comodule structures. Then A is a right H -Galois extension of B and a projective left B -module.*

In particular, the assumption can be verified in the following cases:

1. H is finite dimensional.
2. A is injective as H -comodule, and H has bijective antipode.

Proof First, if β_0 splits as indicated via a map $\alpha: A^* \otimes H^S \rightarrow A^* \otimes A$ with $\beta_0\alpha = id$, then the composition

$$\gamma = \left(H \xrightarrow{\eta^{\otimes H}} A \otimes H \xrightarrow{\alpha} A \otimes A \right)$$

satisfies the assumptions of Proposition 2.4.7.

If A is an injective comodule, and H has bijective antipode, then every Hopf module in ${}_A\mathcal{M}^H$ is an injective comodule by Proposition 2.4.5. Thus the (kernel of the) Hopf module morphism β_0 splits as a comodule map. Finally, if H is finite dimensional, then we take the view that β_0 should split as a surjective H^* -module map. But H^S is projective as H^* -module, and hence $A \otimes H^S$ is projective as well, and thus the map splits. \square

As a corollary, we obtain Schneider's characterization of faithfully flat Hopf-Galois extensions from [44] (and in addition projectivity of such extensions).

Corollary 2.4.9 *Let H be a Hopf algebra with bijective antipode over a base field k , A a right H -comodule algebra, and $B := A^{\text{co}H}$. The following are equivalent:*

1. *The Galois map $A \otimes A \rightarrow A \otimes H$ is onto, and A is injective as H -comodule.*
2. *A is an H -Galois extension of B , and right faithfully flat as B -module.*
3. *A is an H -Galois extension of B , and left faithfully flat as B -module.*

In this case, A is a projective left and right B -module.

Proof We already know from the beginning of this section that $2 \Rightarrow 1$. Assume 1. Then Corollary 2.4.8 implies that A is Galois and a projective left B -module, and that the counit of the adjunction in Lemma 2.3.2 is an isomorphism. By Corollary 2.3.6 it remains to prove that the unit $N \rightarrow (N \otimes_B A)^{\text{co}H}$ is also a bijection for all $N \in \mathcal{M}_B$. But $N \otimes_B A$ is defined by a coequalizer

$$N \otimes B \otimes A \rightrightarrows N \otimes A \rightarrow N \otimes_B A \rightarrow 0,$$

which is a coequalizer in the category \mathcal{M}_A^H . Since every Hopf module is an injective comodule, every short exact sequence in \mathcal{M}_A^H splits colinearly, so the coinvariants functor $\mathcal{M}_A^H \rightarrow \mathcal{M}_B$ is exact, and applying it to the coequalizer above we obtain a coequalizer

$$N \otimes B \otimes A \rightrightarrows N \otimes B \rightarrow (N \otimes_B A)^{\text{co}H}$$

which says that $(N \otimes_B A)^{\text{co}H} \cong N \otimes_B B \cong N$.

The equivalence of 1 and 3 is proved by applying that of 1 and 2 to the H^{op} comodule algebra A^{op} . \square

2.5 Galois extensions as monoidal functors. In this section we prove the characterization of Hopf-Galois extensions as monoidal functors from the category of comodules due to Ulbrich [53, 54]. We are somewhat more general in allowing the invariant subring to be different from the base ring. In this general setting, we have proved one direction of the characterization in [35], but the proof is really no different from Ulbrich's. Some details of the reverse direction (from functors to extensions) are perhaps new. It will turn out that in fact suitably exact *weak* monoidal functors on the category of comodules are the same as comodule algebras, while being monoidal rather than only weak monoidal is related to the Galois condition.

Proposition 2.5.1 *Let H be a bialgebra, and $A \in \mathcal{M}^H$ coflat. If A is an H -comodule algebra, then*

$$\xi: (A \square_H V) \otimes (A \square_H W) \ni (x \otimes v) \otimes (y \otimes w) \mapsto xy \otimes v \otimes w \in A \square_H (V \otimes W)$$

and $\xi_0: k \ni \alpha \mapsto 1 \otimes \alpha \in A \square_H k$ define the structure of a weak monoidal functor on $A \square_H -: {}^H\mathcal{M} \rightarrow \mathcal{M}_k$.

Conversely, every weak monoidal functor structure on $A \square_H -$ has the above form for a unique H -comodule algebra structure on A .

Proof The first claim is easy to check. For the second, given a monoidal functor structure ξ , define multiplication on A as the composition

$$A \otimes A \cong (A \square_H H) \otimes (A \square_H H) \xrightarrow{\xi} A \square_H (H \otimes H) \xrightarrow{A \square_H \nabla} A \square_H H \cong A.$$

By naturality of ξ in its right argument, applied to $\Delta: H \rightarrow {}^*H \otimes H$, we have a commutative diagram

$$\begin{array}{ccc}
(A \square_H H) \otimes (A \square_H H) & \xrightarrow{\xi} & A \square_H (H \otimes H) \\
\downarrow A \square_H H \otimes A \square_H \Delta & & \downarrow A \square_H (H \otimes \Delta) \\
(A \square_H H) \otimes (A \square_H (H \otimes H)) & & A \square_H (H \otimes H \otimes H) \\
\downarrow & & \downarrow \\
(A \square_H H) \otimes (A \square_H H) \otimes H & \xrightarrow{\xi \otimes H} & (A \square_H (H \otimes H)) \otimes H
\end{array}$$

In other words, $\xi: (A \square_H H) \otimes (A \square_H H^*) \rightarrow A \square_H (H \otimes H^*)$ is an H -comodule map with respect to the indicated structures. Similarly (though a little more complicated to write), $\xi: (A \square_H H^*) \otimes (A \square_H H) \rightarrow A \square_H (H^* \otimes H)$ is also colinear, and from both we deduce that $\xi: (A \square_H H^*) \otimes (A \square_H H^*) \rightarrow A \square_H (H^* \otimes H^*)$ is colinear. Hence the multiplication on A is colinear. Associativity of multiplication follows from coherence of ξ , so that A is a comodule algebra. \square

Corollary 2.5.2 *Let H be a bialgebra, A a right H -comodule algebra, and $\iota: B \rightarrow A^{\text{co}H}$ a subalgebra.*

Then for each $V \in {}^H\mathcal{M}$ we have $A \square_H V \in {}_B\mathcal{M}_B$ with bimodule structure induced by that of A (induced in turn by ι). The weak monoidal functor structure in Proposition 2.5.1 induces a weak monoidal functor structure on $A \square_H (-): {}^H\mathcal{M} \rightarrow {}_B\mathcal{M}_B$, which we denote again by

$$\xi: (A \square_H V) \otimes_B (A \square_H W) \ni x \otimes v \otimes y \otimes w \mapsto xy \otimes v \otimes w \in A \square_H (V \otimes W)$$

and $\xi_0: B \ni b \mapsto b \otimes 1 \in A \square_H k$. If A is a (faithfully) coflat H -comodule, the functor is (faithfully) exact

Every exact weak monoidal functor ${}^H\mathcal{M} \rightarrow {}_B\mathcal{M}_B$ commuting with arbitrary direct sums, for a k -algebra B , has this form.

Proof Again, it is not hard to verify that every comodule algebra A and homomorphism ι gives rise to a weak monoidal functor as stated. For the converse, note that a weak monoidal functor ${}^H\mathcal{M} \rightarrow {}_B\mathcal{M}_B$ can be composed with the weak monoidal underlying functor ${}_B\mathcal{M}_B \rightarrow \mathcal{M}_k$ to yield a weak monoidal functor ${}^H\mathcal{M} \rightarrow \mathcal{M}_k$. The latter is exact by assumption, so has the form $V \mapsto A \square_H V$ for some coflat H -comodule A by Lemma 4.3.3, and A is an H -comodule algebra by Proposition 2.5.1. One ingredient of the weak monoidal functor structure that we assume to exist is a B - B -bimodule map $\xi_0: B \rightarrow A \square_H k$ with $\xi_0(1) = 1$, which has the form $\xi_0(b) = \iota(b) \otimes 1$ for some map $\iota: B \rightarrow A^{\text{co}H}$ that also satisfies $\iota(b) = 1$. By coherence of the weak monoidal functor, the left B -module structure of $A \square_H V$, which is also one of the coherence isomorphisms of the monoidal category of B - B -bimodules, is given by

$$B \otimes_B (A \square_H V) \xrightarrow{\xi_0 \otimes id} (A \square_H k) \otimes_B (A \square_H V) \xrightarrow{\xi} A \square_H V.$$

Thus $b \cdot (x \otimes v) = \iota(b)x \otimes v$ holds for all $b \in B$ and $x \otimes v \in A \square_H V$. If we specialize $V = H$ and use the isomorphism $A \square_H H$, we see that ι is an algebra homomorphism, and for general V we see that $A \square_H V$ has the claimed B - B -bimodule structure. \square

Theorem 2.5.3 *Let H be a k -flat Hopf algebra, and B a k -algebra.*

1. *Every exact monoidal functor $\mathcal{F}: {}^H\mathcal{M} \rightarrow {}_B\mathcal{M}_B$ that commutes with arbitrary colimits has the form $\mathcal{F}(V) = A \square_H V$ for some right coflat H -Galois extension A of B , with monoidal functor structure given as in Corollary 2.5.2.*
2. *Assume that A is a right faithfully flat H -Galois extension of B . Then the weak monoidal functor $A \square_H \text{---}$ as in Corollary 2.5.2 is monoidal.*

If we assume that k is a field, and H has bijective antipode, then a Hopf-Galois extension is coflat as H -comodule if and only if it is faithfully flat as right (or left) B -module. Also, if k is arbitrary, then a Hopf-Galois extension of k is faithfully coflat as H -comodule if and only if it is faithfully flat as k -module. Thus we have:

Corollary 2.5.4 *Let H be a Hopf algebra and B a k -algebra. Assume either of the following conditions:*

1. *k is a field and the antipode of H is bijective.*
2. *$B = k$.*

Then Corollary 2.5.2 establishes a bijective correspondence between exact monoidal functors ${}^H\mathcal{M} \rightarrow {}_B\mathcal{M}_B$ and faithfully flat H -Galois extensions of B .

Closing the section, let us give two curious application of the monoidal functor associated to a Galois object.

If H is a Hopf algebra, then any $V \in {}^H\mathcal{M}$ that is a finitely generated projective k -module has a right dual object in the monoidal category ${}^H\mathcal{M}$. Monoidal functors preserve duals. Thus, whenever A is a right faithfully flat H -Galois extension of B , the B -bimodule $A \square_H V$ will have a right dual in the monoidal category of B -bimodules. This in turn means that $A \square_H V$ is finitely generated projective as a left B -module. We have proved:

Corollary 2.5.5 *Let A be a right H -Galois extension of B and a right faithfully flat B -module. Then for every $V \in {}^H\mathcal{M}$ which is a finitely generated projective k -module, the left B -module $A \square_H V$ is finitely generated projective. If H has bijective antipode, the right B -module $A \square_H V$ is also finitely generated projective.*

The corollary (which has other proofs as well) has a conceptual meaning when we think of A as a principal fiber bundle with structure quantum group H . Then $A \square_H V$ is analogous to the module of sections in an associated vector bundle with fiber V , and it is of course good to know that such a module of sections is projective, in keeping with the classical Serre-Swan theorem.

Definition 2.5.6 Let H be a k -flat Hopf algebra, and B a k -algebra. We define $\text{Gal}_B(H)$ to be the set of all isomorphism classes of H -Galois extensions of B that are faithfully flat as right B -modules and (faithfully) flat as k -modules. We write $\text{Gal}(H) = \text{Gal}_B(H)$.

Proposition 2.5.7 *$\text{Gal}_B(\text{---})$ is a contravariant functor. For a Hopf algebra map $f: F \rightarrow H$ between k -flat Hopf algebras, the map $\text{Gal}_B(f): \text{Gal}_B(H) \rightarrow \text{Gal}_B(F)$ maps the isomorphism class of A to that of $A \square_H F$.*

Proof In fact, f defines an exact monoidal functor ${}^F\mathcal{M} \rightarrow {}^H\mathcal{M}$, which composes with the monoidal functor $A \square_H (\text{---}): {}^H\mathcal{M} \rightarrow {}_B\mathcal{M}_B$ defined by A to give the functor $(A \square_H F) \square_F \text{---}$, since $A \square_H V = A \square_H (F \square_F V) \cong (A \square_H F) \square_F V$ by k -flatness of A . This implies that $A \square_H F$ is a right F -Galois extension of B .

It is faithfully flat on the right since A is, and for any left B -module M we have $A \otimes_B (A \square_H F) \otimes_B N \cong ((A \otimes_B A) \square_H F) \otimes_B N \cong (A \otimes H \square_H F) \otimes_B N \cong A \otimes_B N \otimes H$. \square

2.6 Hopf bimodules. Let A be an H -comodule algebra. Since A is an algebra in the monoidal category of H -comodules, we can consider the category of bimodules over A in the monoidal category \mathcal{M}^H . Such a bimodule $M \in {}_A\mathcal{M}_A^H$ is an A -bimodule fulfilling both Hopf module conditions for a Hopf module in \mathcal{M}_A^H and ${}_A\mathcal{M}^H$. By the general theory of modules over algebras in monoidal categories, the category ${}_A\mathcal{M}_A^H$ is a monoidal category with respect to the tensor product over A . Now without further conditions, taking coinvariants gives a *weak* monoidal functor:

Lemma 2.6.1 *Let A be an H -comodule algebra, and let $B \subset A^{\text{co}H}$ be a subalgebra. Then*

$${}_A\mathcal{M}_A^H \ni M \mapsto M^{\text{co}H} \in {}_B\mathcal{M}_B$$

is a weak monoidal functor with structure maps

$$\xi_0: M^{\text{co}H} \otimes_B N^{\text{co}H} \ni m \otimes n \mapsto m \otimes n \in (M \otimes_A N)^{\text{co}H}$$

and $\xi_0: B \rightarrow A^{\text{co}H}$ the inclusion.

The proof is straightforward. The main result of this section is that the functor from the Lemma is monoidal rather than only weak monoidal if and only if A is an H -Galois extension. The precise statement is slightly weaker:

Proposition 2.6.2 *Let H be a Hopf algebra, A a right H -comodule algebra, and $B := A^{\text{co}H}$.*

If A is a left faithfully flat H -Galois extension of B , then the weak monoidal functor from Lemma 2.6.1 is monoidal.

Conversely, if the weak monoidal functor from Lemma 2.6.1 is monoidal, then the counit of the adjunction 2.3.2 is an isomorphism, and in particular, A is an H -Galois extension of B .

Proof If A is a left faithfully flat H -Galois extension of B , then ξ is an isomorphism if and only if $\xi \otimes_B A$ is. But via the isomorphisms

$$M \otimes_A N \cong M^{\text{co}H} \otimes_B A \otimes_A N \cong M^{\text{co}H} \otimes_B N \cong M^{\text{co}H} \otimes_B N^{\text{co}H} \otimes_B A$$

and $(M \otimes_A N)^{\text{co}H} \otimes_B A \cong M \otimes_A N$, the map $\xi \otimes_B A: M^{\text{co}H} \otimes_B N^{\text{co}H} \otimes_B A \rightarrow (M \otimes_A N)^{\text{co}H} \otimes_B A$ identifies with the identity on $M \otimes_A N$.

Conversely, if ξ is an isomorphism, we can specialize $N := \cdot A \cdot \otimes \cdot H \cdot \in {}_A\mathcal{M}_A^H$. We have $N^{\text{co}H} \cong A$. In ${}_A\mathcal{M}_A^H$ we have an isomorphism

$$\cdot A \cdot \otimes H \cdot \cong \cdot A \cdot \otimes H \cdot; a \otimes h \mapsto a_{(0)} \otimes a_{(1)}h.$$

Thus we have an isomorphism

$$M \otimes_A N \cong M \otimes_A (\cdot A \cdot \otimes H \cdot) \cong M \cdot \otimes H \cdot; m \otimes a \otimes h \mapsto ma_{(0)} \otimes S(a_{(1)})h.$$

composing with $M \cdot \otimes H \cdot \ni m \otimes h \mapsto m_{(0)} \otimes m_{(1)}h \in M \otimes H \cdot$ yields the isomorphism

$$M \otimes_A N \cong M \otimes H \cdot; m \otimes a \otimes h \mapsto m_{(0)}a \otimes m_{(1)}h.$$

Thus we find that

$$M^{\text{co}H} \otimes_B A \cong M^{\text{co}H} \otimes_B N^{\text{co}H} \xrightarrow{\xi} (M \otimes_A N)^{\text{co}H} \cong (M \otimes H \cdot)^{\text{co}H} \cong M$$

maps $m \otimes a$ to ma , hence is the adjunction counit in question. \square

2.7 Reduction. We have already seen that $\text{Gal}_B(-)$ is a functor. In particular, we have a map $\text{Gal}_B(Q) \rightarrow \text{Gal}_B(H)$ for any (suitable) quotient Hopf algebra Q of H . In this section we will be concerned with the image and fibers of this map. The question has a geometric interpretation when we think of Galois extensions as principal fiber bundles: It is then the question under what circumstances a principal bundle with structure group G can be reduced to a principal bundle whose structure group is a prescribed subgroup of G .

The results in this section were proved first in [37] for the case of conormal quotients Q (i.e. normal subgroups, when we think of principal homogeneous spaces). The general case was obtained in [20, 21]. The proof we give here was essentially given in [43]; we rewrite it here with (yet) more emphasis on its background in the theory of algebras in monoidal categories. We begin with a Theorem of Takeuchi [49] on Hopf modules for a quotient of a Hopf algebra. We prove a special case in a new way here, which we do not claim to be particularly natural, but which only uses category equivalences that we have already proved above.

Theorem 2.7.1 *Let H be a k -flat Hopf algebra, and $H \rightarrow Q$ a quotient Hopf algebra of H which is also k -flat and has bijective antipode. Assume that H is a left Q -Galois extension of $K := {}^{\text{co}Q}H$, and faithfully flat as left as well as right K -module.*

Then $\mathcal{M}_K^H \ni M \mapsto M/MK^+ \in \mathcal{M}^Q$ is a category equivalence. The inverse equivalence maps $N \in \mathcal{M}^Q$ to $N \square_Q H$ with the K -module and H -comodule structures induced by those of H .

Remark 2.7.2 As we learned in Corollary 2.4.9, our list of requirements on the quotient $H \rightarrow Q$ is fulfilled if k is a field, Q has bijective antipode, and H is a coflat left Q -comodule.

Proof By the structure theorem for Hopf modules over a Hopf-Galois extension, we have an equivalence $\mathcal{F}: \mathcal{M}_K \rightarrow {}^Q\mathcal{M}_H$ given by $\mathcal{F}(N) = N \otimes_K H$, with quasi-inverse $\mathcal{F}(M) = {}^{\text{co}Q}M$. We claim that \mathcal{F} induces an equivalence $\hat{\mathcal{F}}: \mathcal{M}_K^H \rightarrow {}^Q\mathcal{M}_H^H$. Indeed, if $N \in \mathcal{M}_K^H$, then $N \otimes_K H$ is an object of ${}^Q\mathcal{M}_H^H$ when endowed with the diagonal right H -module structure (which is well-defined since K is an H -comodule subalgebra of H). Conversely, if $M \in {}^Q\mathcal{M}_H^H$, then ${}^{\text{co}Q}M$ is a right H -subcomodule of M and in this way a Hopf module in \mathcal{M}_K^H . It is straightforward to check that the adjunction morphisms for \mathcal{F} and \mathcal{F}^{-1} are compatible with these additional structures.

Next, we have the category equivalence $\mathcal{M}_H^H \cong \mathcal{M}_k$, which induces an equivalence $\mathcal{G}: {}^Q\mathcal{M} \rightarrow {}^Q\mathcal{M}_H^H$. Indeed, if $V \in {}^Q\mathcal{M}$, then $V \otimes H \in {}^Q\mathcal{M}_H^H$ with the codiagonal left Q -comodule structure, and conversely, if $M \in {}^Q\mathcal{M}_H^H$, then $M^{\text{co}H}$ is a left Q -subcomodule of M . Now consider the composition

$$\mathcal{T} := \left(\mathcal{M}^Q \rightarrow {}^Q\mathcal{M} \xrightarrow{\mathcal{G}} {}^Q\mathcal{M}_H^H \xrightarrow{\hat{\mathcal{F}}^{-1}} \mathcal{M}_K^H \right)$$

where the first functor is induced by the inverse of the antipode. We have $\mathcal{T}(V) = {}^{\text{co}Q}(V^{S^{-1}} \otimes H) = V \square_Q H$. We leave it to the reader to check that the module and comodule structure of $\mathcal{T}(V)$ are indeed those induced by H .

Since it is, finally, easy to check that the $\mathcal{M}_K^H \ni M \rightarrow M/MK^+ \in \mathcal{M}^Q$ is left adjoint to \mathcal{T} , it is also its quasi-inverse. \square

Let H and Q be k -flat Hopf algebras, and $\nu: H \rightarrow Q$ a Hopf algebra map. Then $K := {}^{\text{co}}Q H$ is stable under the right adjoint action of H on itself defined by $x \leftarrow h = S(h_{(1)})xh_{(2)}$, since for $x \in K$ we have $\nu((x \leftarrow h)_{(1)}) \otimes (x \leftarrow h)_{(2)} = \nu(S(h_{(2)})x_{(1)}h_{(3)}) \otimes S(h_{(1)})x_{(2)}h_{(4)} = \nu(S(h_{(2)})h_{(3)}) \otimes S(h_{(1)})xh_{(4)} = 1 \otimes S(h_{(1)})xh_{(2)}$ for all $h \in H$. Thus K is a subalgebra of the (commutative) algebra H in the category \mathcal{YD}_H^H of right-right H -Yetter-Drinfeld modules, which in turn is the center of the monoidal category \mathcal{M}^H of right H -comodules.

As a corollary, the category \mathcal{M}_K^H is equivalent to the monoidal subcategory $\mathcal{S} \subset {}_K\mathcal{M}_K^H$ of symmetric bimodules in \mathcal{M}^H , that is, the category of those $M \in {}_K\mathcal{M}_K^H$ for which $xm = m_{(0)}(x \leftarrow m_{(1)})$ holds for all $m \in M$ and $x \in K$. The equivalence is induced by the underlying functor ${}_K\mathcal{M}_K^H \rightarrow \mathcal{M}_K^H$. Since now the source and target of the equivalence in Theorem 2.7.1 are monoidal functors, the following Theorem answers an obvious question:

Theorem 2.7.3 *The category equivalence from Theorem 2.7.1 is a monoidal category equivalence with respect to the isomorphisms*

$$\xi: (V \square_Q H) \otimes_K (W \square_Q H) \ni v \otimes g \otimes w \otimes h \mapsto v \otimes w \otimes gh \in (V \otimes W) \square_Q H$$

Proof We have already seen (with switched sides) in Section 2.5 that ξ makes $(-) \square_Q H: \mathcal{M}^Q \rightarrow {}_K\mathcal{M}_K^H$ a monoidal functor. Quite obviously, $V \square_Q H$ has the structure of a right H -comodule in such a way that $V \square_Q H \in {}_K\mathcal{M}_K^H$, and ξ is an H -comodule map. Thus, we have a monoidal functor $(-) \square_Q H: \mathcal{M}^Q \rightarrow {}_K\mathcal{M}_K^H$. For $t = v \otimes h \in V \square_Q H$ we have $xt = v \otimes xh = v \otimes h_{(1)}(x \leftarrow h_{(2)}) = t_{(0)}(x \leftarrow t_{(1)})$, so that the monoidal functor $(-) \square_Q H$ takes values in the subcategory \mathcal{S} . Observe, finally, that it composes with the underlying functor to \mathcal{M}_K^H to give the equivalence of categories from Theorem 2.7.1. From the commutative triangle

$$\begin{array}{ccc} \mathcal{M}^Q & \xrightarrow{(-) \square_Q H} & \mathcal{S} \\ & \searrow & \swarrow u \\ & & \mathcal{M}_K^H \\ & \swarrow & \searrow \\ & & \mathcal{M}_K^H \end{array}$$

of functors, in which we already know the slanted arrows to be equivalences, we deduce that the top arrow is an equivalence. \square

Corollary 2.7.4 *Assume the hypotheses of Theorem 2.7.1.*

The categories of right Q -comodule algebras, and of algebras in the category \mathcal{M}_K^H are equivalent. The latter consists of pairs (A, f) in which A is a right H -comodule algebra, and $f: K \rightarrow A$ is a right H -comodule algebra map satisfying $f(x)a = a_{(0)}f(x \leftarrow a_{(1)})$ for all $x \in K$ and $a \in A$.

Suppose given an algebra $A \in \mathcal{M}_K^H$, corresponding to an algebra $\bar{A} \in \mathcal{M}^Q$. Then the categories of right A -modules in \mathcal{M}_K^H and of right \bar{A} -modules in \mathcal{M}^Q are equivalent. The former is the category of right A -modules in \mathcal{M}_K^H , so we have a

commutative diagram of functors

$$\begin{array}{ccc}
 \mathcal{M}_A^Q & \xrightarrow{(-)\square_Q H} & \mathcal{M}_A^H \\
 & \searrow^{(-)^{\text{co } H}} & \swarrow_{(-)^{\text{co } Q}} \\
 & & \mathcal{M}_B
 \end{array}$$

where $B = A^{\text{co } H} \cong \overline{A}^{\text{co } Q}$.

Corollary 2.7.5 *Assume the hypotheses of Theorem 2.7.1. Then we have a bijection between*

1. *isomorphism classes of left faithfully flat Q -Galois extensions of B , and*
2. *equivalences of pairs (A, f) in which A is a left faithfully flat H -Galois extension of B , and $f: K \rightarrow A^B$ is a homomorphism of algebras in \mathcal{YD}_H^H . Here, two pairs (A, f) and (A', f') are equivalent if there is a B -linear H -comodule algebra map $t: A \rightarrow A'$ such that $tf = f'$.*

Proof We know that Q -comodule algebras \overline{A} correspond to isomorphism classes of pairs (A, f) in which A is an H -comodule algebra and $f: K \rightarrow A$ an H -comodule algebra map which is central in the sense of Definition 4.2.1. Since faithfully flat Galois extensions are characterized by the structure theorem for Hopf modules, see Corollary 2.3.6, the diagram in Corollary 2.7.4 shows that \overline{A} is faithfully flat Q -Galois if and only if A is faithfully flat H -Galois. But if A is faithfully flat H -Galois, then every central H -comodule algebra map factors through a Yetter-Drinfeld module algebra map to A^B by Lemma 2.1.9. \square

The preceding corollary can be restated as follows:

Corollary 2.7.6 *Assume the hypotheses of Theorem 2.7.1. Consider the map $\pi: \text{Gal}_B(Q) \rightarrow \text{Gal}_B(H)$ given by $\pi(\overline{A}) = A \square_Q H$, and let $A \in \text{Gal}_B(H)$. Then*

$$\pi^{-1}(A) \cong \text{Alg}_{-H}^{-H}(K, A^B) / \text{Aut}_B^H(A),$$

where $\text{Aut}_B^H(A)$ acts on $\text{Alg}_{-H}^{-H}(K, A^B)$ by composition.

2.8 Hopf Galois extensions without Hopf algebras. Cyril Grunspan [19] has revived an idea that appears to have been known in the case of commutative Hopf-Galois extensions (or torsors) for a long time, going back to a paper of Reinhold Baer [1]: It is possible to write down axioms characterizing a Hopf-Galois extension without mentioning a Hopf algebra.

This approach to (noncommutative) Hopf-Galois extensions begins in [19] with the definition of a quantum torsor (an algebra with certain additional structures) and the proof that every quantum torsor gives rise to two Hopf algebras over which it is a bi-Galois extension of the base field. The converse was proved in [41]: Every Hopf-Galois extension of the base field is a quantum torsor in the sense of Grunspan. Then the axioms of a quantum torsor were simplified in [42] by showing that a key ingredient of Grunspan's definition (a certain endomorphism of the torsor) is actually not needed to show that a torsor is a Galois object. The simplified version of the torsor axioms admits a generalization to general Galois extensions (not only of the base ring or field).

Definition 2.8.1 Let B be a k -algebra, and $B \subset T$ an algebra extension, with T a faithfully flat k -module. The centralizer $(T \otimes_B T)^B$ of B in the (obvious) B - B -bimodule $T \otimes_B T$ is an algebra by $(x \otimes y)(a \otimes b) = ax \otimes yb$ for $x \otimes y, a \otimes b \in (T \otimes_B T)^B$.

A B -torsor structure on T is an algebra map $\mu: T \rightarrow T \otimes (T \otimes_B T)^B$; we denote by $\mu_0: T \rightarrow T \otimes T \otimes_B T$ the induced map, and write $\mu_0(x) = x^{(1)} \otimes x^{(2)} \otimes x^{(3)}$.

The torsor structure is required to fulfill the following axioms:

$$x^{(1)}x^{(2)} \otimes x^{(3)} = 1 \otimes x \in T \otimes_B T \quad (2.8.1)$$

$$x^{(1)} \otimes x^{(2)}x^{(3)} = x \otimes 1 \in T \otimes T \quad (2.8.2)$$

$$\mu(b) = b \otimes 1 \otimes 1 \quad \forall b \in B \quad (2.8.3)$$

$$\mu(x^{(1)}) \otimes x^{(2)} \otimes x^{(3)} = x^{(1)} \otimes x^{(2)} \otimes \mu(x^{(3)}) \in T \otimes T \otimes_B T \otimes_B T \quad (2.8.4)$$

Note that (2.8.4) makes sense since μ is a left B -module map by (2.8.3).

Remark 2.8.2 If $B = k$, then the torsor axioms simplify as follows: They now assume the existence of an algebra map $\mu: T \rightarrow T \otimes T^{\text{op}} \otimes T$ such that the diagrams

$$\begin{array}{ccc} T & \xrightarrow{\mu} & T \otimes T^{\text{op}} \otimes T \\ \downarrow \mu & & \downarrow T \otimes T^{\text{op}} \otimes \mu \\ T \otimes T^{\text{op}} \otimes T & \xrightarrow{\mu \otimes T^{\text{op}} \otimes T} & T \otimes T^{\text{op}} \otimes T \otimes T^{\text{op}} \otimes T \\ & & \downarrow \mu \\ & & T \\ & \swarrow T \otimes \eta & \searrow \eta \otimes T \\ T \otimes T & \xleftarrow{T \otimes \nabla} & T \otimes T \otimes T \xrightarrow{\nabla \otimes T} T \otimes T \end{array}$$

commute.

The key observation is now that a torsor provides a descent data. Here we use left descent data, i.e. certain S -linear maps $\theta: M \rightarrow S \otimes_R M$ for a ring extension $R \subset S$ and a left S -module M , as opposed to the right descent data in Section 4.5. For a left descent data $\theta: M \rightarrow S \otimes_R M$ from S to R on a left S -module M we will write ${}^\theta M := \{m \in M \mid \theta(m) = 1 \otimes m\}$.

Lemma 2.8.3 *Let T be a B -torsor. Then a descent data D from T to k on $T \otimes_B T$ is given by $D(x \otimes y) = xy^{(1)} \otimes y^{(2)} \otimes y^{(3)}$. It satisfies $(T \otimes D)\mu(x) = x^{(1)} \otimes 1 \otimes x^{(2)} \otimes x^{(3)}$.*

Proof Left T -linearity of D is obvious. We have

$$\begin{aligned} (T \otimes D)\mu(x) &= x^{(1)} \otimes D(x^{(2)} \otimes x^{(3)}) \\ &= x^{(1)} \otimes (\nabla \otimes T \otimes T)(x^{(2)} \otimes \mu(x^{(3)})) \\ &= (T \otimes \nabla \otimes T \otimes T)(\mu(x^{(1)}) \otimes x^{(2)} \otimes x^{(3)}) \\ &= x^{(1)} \otimes 1 \otimes x^{(2)} \otimes x^{(3)} \end{aligned}$$

and thus

$$\begin{aligned} (T \otimes D)D(x \otimes y) &= xy^{(1)} \otimes D(y^{(2)} \otimes y^{(3)}) \\ &= xy^{(1)} \otimes 1 \otimes y^{(2)} \otimes y^{(3)} \\ &= (T \otimes \eta \otimes T \otimes T)D(x \otimes y). \end{aligned}$$

Finally $(\nabla \otimes T \otimes T)D(x \otimes y) = xy^{(1)}y^{(2)} \otimes y^{(3)} = x \otimes y$. \square

Note that $D(T \otimes_B T) \subset T \otimes (T \otimes_B T)^B$. Since T is faithfully flat over k , then faithfully flat descent implies that ${}^D(T \otimes_B T) \subset (T \otimes_B T)^B$.

Theorem 2.8.4 *Let T be a B -torsor, and assume that T is a faithfully flat right B -module.*

Then $H := {}^D(T \otimes_B T)$ is a k -flat Hopf algebra. The algebra structure is that of a subalgebra of $(T \otimes_B T)^B$, the comultiplication and counit are given by

$$\begin{aligned} \Delta(x \otimes y) &= x \otimes y^{(1)} \otimes y^{(2)} \otimes y^{(3)}, \\ \varepsilon(x \otimes y) &= xy \end{aligned}$$

for $x \otimes y \in H$. The algebra T is an H -Galois extension of B under the coaction $\delta: T \rightarrow T \otimes H$ given by $\delta(x) = \mu(x)$.

Proof H is a subalgebra of $(T \otimes_B T)^B$ since for $x \otimes y, a \otimes b \in H$ we have

$$\begin{aligned} D((x \otimes y)(a \otimes b)) &= D(ax \otimes yb) \\ &= ax(yb)^{(1)} \otimes (yb)^{(2)} \otimes (yb)^{(3)} \\ &= axy^{(1)}b^{(1)} \otimes b^{(2)}y^{(2)} \otimes y^{(3)}b^{(3)} \\ &= ab^{(1)} \otimes b^{(2)}x \otimes yb^{(3)} \\ &= 1 \otimes ax \otimes yb \\ &= 1 \otimes (x \otimes y)(a \otimes b). \end{aligned}$$

To see that the coaction δ is well-defined, we have to check that the image of μ is contained in $T \otimes H$, which is, by faithful flatness of T , the equalizer of

$$T \otimes T \otimes_B T \begin{array}{c} \xrightarrow{T \otimes D} \\ \xrightarrow{T \otimes \eta \otimes T \otimes_B T} \end{array} T \otimes T \otimes T \otimes_B T.$$

But $(T \otimes D)\mu(x) = (T \otimes \eta \otimes T \otimes_B T)\mu(x)$ was shown in Lemma 2.8.3. Since μ is an algebra map, so is the coaction δ , for which we employ the usual Sweedler notation $\delta(x) = x_{(0)} \otimes x_{(1)}$. Note that (2.8.3) implies that $\delta(b) = b \otimes 1$ for all $b \in B$; in other words, δ is left B -linear.

The Galois map $\beta: T \otimes_B T \rightarrow T \otimes H$ for the coaction δ is given by $\beta(x \otimes y) = xy_{(0)} \otimes y_{(1)} = xy^{(1)} \otimes y^{(2)} \otimes y^{(3)} = D(x \otimes y)$. Thus it is an isomorphism by faithfully flat descent, Theorem 4.5.2. It follows that H is faithfully flat over k .

Since δ is left B -linear,

$$\Delta_0: T \otimes_B T \ni x \otimes y \mapsto x \otimes y^{(1)} \otimes y^{(2)} \otimes y^{(3)} \in T \otimes_B T \otimes H$$

is well-defined. To prove that Δ is well-defined, we need to check that $\Delta_0(H)$ is contained in $H \otimes H$, which, by faithful flatness of H , is the equalizer of

$$T \otimes_B T \otimes H \begin{array}{c} \xrightarrow{D \otimes H} \\ \xrightarrow{\eta \otimes T \otimes T \otimes H} \end{array} T \otimes T \otimes_B T \otimes H.$$

Now for $x \otimes y \in H$ we have

$$\begin{aligned}
(D \otimes H)\Delta_0(x \otimes y) &= (D \otimes H)(x \otimes y^{(1)} \otimes y^{(2)} \otimes y^{(3)}) \\
&= xy^{(1)(1)} \otimes y^{(1)(2)} \otimes y^{(1)(3)} \otimes y^{(2)} \otimes y^{(3)} \\
&= xy^{(1)} \otimes y^{(2)} \otimes \mu(y^{(3)}) \\
&= (T \otimes T \otimes \mu)D(x \otimes y) \\
&= (T \otimes T \otimes \mu)(1 \otimes x \otimes y) \\
&= 1 \otimes \Delta_0(x \otimes y)
\end{aligned}$$

Δ is an algebra map since μ is, and coassociativity follows from the coassociativity axiom of the torsor T .

For $x \otimes y \in H$ we have $xy \otimes 1 = xy^{(1)} \otimes y^{(2)}y^{(3)} = 1 \otimes xy$, whence $xy \in k$ by faithful flatness of T . Thus, ε is well-defined. It is straightforward to check that ε is an algebra map, that it is a counit for Δ , and that the coaction δ is counital. Thus, H is a bialgebra.

We may now write the condition $\delta(b) = b \otimes 1$ for $b \in B$ simply as $B \subset T^{\text{co}H}$. Conversely, $x \in T^{\text{co}H}$ implies $x \otimes 1 = x^{(1)}x^{(2)} \otimes x^{(3)} = 1 \otimes x \in T \otimes_B T$, and thus $x \in B$ by faithful flatness of T as a B -module. Since we have already seen that the Galois map for the H -extension $B \subset T$ is bijective, T is an H -Galois extension of B , and from Lemma 2.1.5 we deduce that H is a Hopf algebra. \square

Lemma 2.8.5 *Let H be a k -faithfully flat Hopf algebra, and let T be a right faithfully flat H -Galois extension of $B \subset T$. Then T is a B -torsor with torsor structure*

$$\mu: T \ni x \mapsto x_{(0)} \otimes x_{(1)}^{[1]} \otimes x_{(1)}^{[2]} \in T \otimes (T \otimes_B T)^B,$$

where $h^{[1]} \otimes h^{[2]} = \beta^{-1}(1 \otimes h) \in T \otimes_B T$, with $\beta: T \otimes_B T \rightarrow T \otimes H$ the Galois map.

3 Hopf-bi-Galois theory

3.1 The left Hopf algebra. Let A be a faithfully flat H -Galois object. Then A is a torsor by Lemma 2.8.5. By the left-right switched version of Theorem 2.8.4, there exists a Hopf algebra $L := L(A, H)$ such that A is a left L -Galois extension of k . Moreover, since the torsor structure $\mu: A \rightarrow A \otimes A^{\text{op}} \otimes A$ is right H -colinear, we see that A is an L - H -bicomodule.

Definition 3.1.1 An L - H -Bi-Galois object is a k -faithfully flat L - H -bicomodule algebra A which is simultaneously a left L -Galois object and a right H -Galois object.

We have seen that every right H -Galois object can be endowed with a left L -comodule algebra structure making it an L - H -Bi-Galois object. We shall prove uniqueness by providing a universal property shared by every L that makes a given H -Galois object into an L - H -bi-Galois object.

Proposition 3.1.2 *Let H and L be k -flat Hopf algebras, and A an L - H -bi-Galois object.*

Then for all $n \in \mathbb{N}$ and k -modules V, W we have a bijection

$$\Phi := \Phi_{V, W, n}: \text{Hom}(V \otimes L^{\otimes n}, W) \cong \text{Hom}^{-H}(V \otimes A^{\otimes n}, W \otimes A)$$

(where $A^{\otimes n}$ carries the codiagonal comodule structure), given by $\Phi(f)(v \otimes x_1 \otimes \dots \otimes x_n) = f(v \otimes x_{1(-1)} \otimes \dots \otimes x_{n(-1)}) \otimes x_{1(0)} \cdots x_{n(0)}$.

In particular, for every k -module we have the universal property that every right H -colinear map $\phi: A \rightarrow W \otimes A$ factors uniquely in the form $\phi = (f \otimes A)\delta_\ell$ as in the diagram

$$\begin{array}{ccc} A & \xrightarrow{\delta_\ell} & L \otimes A \\ & \searrow \phi & \downarrow f \otimes A \\ & & W \otimes A \end{array} \quad \begin{array}{c} L \\ \downarrow f \\ W \end{array}$$

Proof Note first that the left Galois map $\beta_\ell: A^* \otimes A^* \rightarrow L \otimes A^*$ is evidently a map of Hopf modules in \mathcal{M}_A^H with the indicated structures. We deduce that for any $M \in \mathcal{M}_A^H$ we have

$$M^* \otimes L^{\otimes m} \otimes A^* \cong M^* \otimes L^{\otimes(m-1)} \otimes A^* \otimes A^* \cong M^* \otimes A^* \otimes L^{\otimes(m-1)} \otimes A^*$$

in \mathcal{M}_A^H , and hence by induction

$$V \otimes L^{\otimes n} \otimes A^* \cong V \otimes (A^*)^{\otimes n} \otimes A^* \in \mathcal{M}_A^H$$

We can now use the structure theorem for Hopf modules, Corollary 2.3.6, to compute

$$\begin{aligned} \text{Hom}(V \otimes L^{\otimes n}, W) &\cong \text{Hom}_{-A}^{-H}(V \otimes L^{\otimes n} \otimes A^*, W \otimes A^*) \\ &\cong \text{Hom}_{-A}^{-H}(V \otimes (A^*)^{\otimes n} \otimes A^*, W \otimes A^*) \cong \text{Hom}^{-H}(V \otimes A^{\otimes n}, W \otimes A). \end{aligned}$$

We leave it to the reader to verify that the bijection has the claimed form. \square

Corollary 3.1.3 *Let A be an L - H -bi-Galois object, B a k -module, $f: L \rightarrow B$, and $\lambda = \Phi(f): A \rightarrow B \otimes A$.*

1. *Assume B is a coalgebra. Then f is a coalgebra map if and only if λ is a comodule structure.*
2. *Assume B is an algebra. Then f is an algebra map if and only if λ is.*
3. *In particular, assume B is a bialgebra. Then f is a bialgebra map if and only if λ is a comodule algebra structure. In particular, the bialgebra L in an L - H -bi-Galois object is uniquely determined by the H -Galois object A .*

Proof We have $\Delta f = (f \otimes f)\Delta: L \rightarrow B \otimes B$ if and only if $\Phi(\Delta f) = \Phi((f \otimes f)\Delta): A \rightarrow B \otimes B \otimes A$. But $\Phi(\Delta f)(a) = (\Delta \otimes A)\lambda$, and $\Phi((f \otimes f)\Delta)(a) = (f \otimes f)\Delta(a_{(-1)}) \otimes a_{(0)} = f(a_{(-2)}) \otimes f(a_{(-1)}) \otimes a_{(0)} = f(a_{(-1)}) \otimes \lambda(a_{(0)}) = (B \otimes \lambda)\lambda(a)$, proving (1).

We have $\nabla(f \otimes f) = f\nabla: L \otimes L \rightarrow B$ if and only if $\Phi(\nabla(f \otimes f)) = \Phi(f\nabla): A \otimes A \rightarrow B \otimes A$. But $\Phi(\nabla(f \otimes f))(x \otimes y) = f(x_{(-1)})f(y_{(-1)}) \otimes x_{(0)}y_{(0)} = \lambda(x)\lambda(y)$ and $\Phi(f\nabla)(x \otimes y) = f(x_{(-1)}y_{(-1)}) \otimes x_{(0)}y_{(0)} = \lambda(xy)$, proving (2).

(3) is simply a combination of (1) and (2), since L as a bialgebra is uniquely determined once it fulfills a universal property for bialgebra maps. \square

Corollary 3.1.4 *Let A be an L - H -bi-Galois object. Then*

$$\text{Alg}(L, k) \ni \varphi \mapsto (a \mapsto \varphi(a_{(-1)})a_{(0)}) \in \text{Aut}^{-H}(A)$$

is an isomorphism from the group of algebra maps from L to k (i.e. the group of grouplikes of L^ if L is finitely generated projective) to the group of H -colinear algebra automorphisms of A .*

If H is cocommutative, then every H -Galois object is trivially an H - H -bi-Galois object, so:

Corollary 3.1.5 *If H is cocommutative and A is an L - H -bi-Galois object, then $L \cong H$.*

It is also obvious that $L(H, H) = H$. There is a more general important case in which $L(A, H)$ can be computed in some sense (see below, though), namely that of cleft extensions:

Proposition 3.1.6 *Let $A = k \#_{\sigma} H$ be a crossed product with invertible cocycle σ . Then $L(A, H) = H$ as coalgebras, while multiplication in $L(A, H)$ is given by*

$$g \cdot h = \sigma(g_{(1)} \otimes h_{(1)})g_{(2)}h_{(2)}\sigma^{-1}(g_{(3)} \otimes h_{(3)}).$$

We will say that $L(A, H) := H^{\sigma}$ is a cocycle double twist of H . The construction of a cocycle double twist is dual to the construction of a Drinfeld twist [17], and was considered by Doi [14]. We have said already that the isomorphism $L(k \#_{\sigma} H, H) = H^{\sigma}$ computes the left Hopf algebra in case of cleft extensions in some sense. In applications, this may rather be read backwards: Cocycles in the non-cocommutative case are not easy to compute for lack of a cohomological interpretation, while it may be easier to guess a left Hopf algebra from generators and relations of A . In this sense the isomorphism may be used to compute the Hopf algebra H^{σ} helped by the left Hopf algebra construction. This is quite important in the applications we will cite in Section 3.2.

We will give a different proof from that in [33] of Proposition 3.1.6. It has the advantage not to use the fact that H^{σ} is a Hopf algebra — checking the existence of an antipode is in fact one of the more unpleasant parts of the construction.

Proof of Proposition 3.1.6 We will not check here that H^{σ} is a bialgebra. Identify $A = k \#_{\sigma} H = H$, with multiplication $g \circ h = \sigma(g_{(1)} \otimes h_{(1)})g_{(2)}h_{(2)}$. Then it is straightforward to verify that comultiplication in H is an H^{σ} -comodule algebra structure $A \rightarrow H^{\sigma} \otimes A$ which, of course, makes A an H^{σ} - H -bicomodule algebra. One may now finish the proof by appealing to Lemma 3.2.5 below, but we will stay more elementary. We shall verify that H^{σ} fulfills the universal property of $L(A, H)$. Of course it does so as a coalgebra, since the left coaction is just the comultiplication of H . Thus a B - H -bicomodule algebra structure $\lambda: A \rightarrow B \otimes A$ gives rise to a unique coalgebra map $f: H^{\sigma} \rightarrow B$ by $f(h) = (B \otimes \varepsilon)\lambda(h) = h_{(-1)}\varepsilon(h_{(0)})$. We have to check that f is an algebra map:

$$\begin{aligned} f(g \cdot h) &= \sigma(g_{(1)} \otimes h_{(1)})f(g_{(2)}h_{(2)})\sigma^{-1}(g_{(3)} \otimes h_{(3)}) = f(g_{(1)} \circ h_{(1)})\sigma^{-1}(g_{(2)} \otimes h_{(2)}) \\ &= (B \otimes \varepsilon)(\lambda(g_{(1)})\lambda(h_{(1)}))\sigma^{-1}(g_{(2)} \otimes h_{(2)}) \\ &= g_{(1)(-1)}h_{(1)(-1)}\varepsilon(g_{(1)(0)} \circ h_{(1)(0)})\sigma^{-1}(g_{(2)} \otimes h_{(2)}) \\ &= g_{(-1)}h_{(-1)}\varepsilon(g_{(0)(1)} \circ g_{(0)(1)})\sigma^{-1}(g_{(0)(2)} \otimes h_{(0)(2)}) = g_{(-1)}h_{(-1)}\varepsilon(g_{(0)} \cdot h_{(0)}) \\ &= g_{(-1)}\varepsilon(g_{(0)})h_{(-1)}\varepsilon(h_{(0)}) = f(g)f(h) \quad \square \end{aligned}$$

Remark 3.1.7 Let A be an L - H -bi-Galois object. The left Galois map $\beta_{\ell}: A^{\bullet} \otimes A^{\bullet} \rightarrow L \otimes A^{\bullet}$ is right H -colinear as indicated, and thus induces an isomorphism $(A \otimes A)^{\text{co}H} \cong (L \otimes A)^{\text{co}H} \cong L$, where the coinvariants of $A \otimes A$ are taken with respect to the codiagonal comodule structure. Let us check that the isomorphism is an algebra map to a subalgebra of $A \otimes A^{\text{op}}$: If $x \otimes y, x' \otimes y' \in A \otimes A$

are such that $x_{(-1)} \otimes x_{(0)}y = \ell \otimes 1$ and $x'_{(-1)} \otimes x'_{(0)}y = \ell' \otimes 1$ for $\ell, \ell' \in L$, then $\beta_\ell(xx' \otimes y'y) = x_{(-1)}x'_{(-1)} \otimes x_{(0)}x'_{(0)}y'y = x_{(-1)}\ell' \otimes x_{(0)}y = \ell\ell' \otimes 1$.

3.2 Monoidal equivalences and the groupoid of bi-Galois objects.

Let H be a Hopf algebra, and A an L - H -bi-Galois object. Then the monoidal functor $(A \square_H _, \xi)$ considered in Section 2.5 also defines a monoidal functor $A \square_H _ : {}^H\mathcal{M} \rightarrow {}^L\mathcal{M}$. If B is an H - R -bi-Galois object, then $A \square_H B$ is an L - H -bicomodule algebra, and since the functor ${}^R\mathcal{M} \ni V \mapsto (A \square_H B) \square_R V \in {}^L\mathcal{M}$ is the composition of the two monoidal functors $(B \square_R _)$ and $A \square_H _$, it is itself monoidal, so that $A \square_H B$ is an R -Galois object by Corollary 2.5.4. By symmetric arguments, $A \square_H B$ is also a left L -Galois and hence an L - R -bi-Galois object. Thus, without further work, we obtain:

Corollary 3.2.1 *k -flat Hopf algebras form a category \underline{BiGal} when we define a morphism from a Hopf algebra H to a Hopf algebra L to be an isomorphism class of L - H -bi-Galois objects, and if we define the composition of bi-Galois objects as their cotensor product.*

On the other hand we can define a category whose objects are Hopf algebras, and in which a morphism from H to L is an isomorphism class of monoidal functors ${}^H\mathcal{M} \rightarrow {}^L\mathcal{M}$.

A functor from the former category to the latter is described by assigning to an L - H -bi-Galois object A the functor $A \square_H _ : {}^H\mathcal{M} \rightarrow {}^L\mathcal{M}$.

The purpose of the Corollary was to collect what we can deduce without further effort from our preceding results. The following Theorem gives the full information:

- Theorem 3.2.2**
1. *The category \underline{BiGal} is a groupoid; that is, for every L - H -bi-Galois object A there is an H - L -bi-Galois object A^{-1} such that $A \square_H A^{-1} \cong L$ as L - L -bicomodule algebras and $A^{-1} \square_L A \cong H$ as H - H -bicomodule algebras.*
 2. *The category \underline{BiGal} is equivalent to the category whose objects are all k -flat Hopf algebras, and in which a morphism from H to L is an isomorphism class of monoidal category equivalences ${}^H\mathcal{M} \rightarrow {}^L\mathcal{M}$.*

If k is a field, there is a short conceptual proof for the Theorem, in which the second claim is proved first, and the first is an obvious consequence. If k is arbitrary, there does not seem to be a way around proving the first claim first. This turns out to be much easier if we assume all antipodes to be bijective. We will sketch all approaches below, but we shall comment first on the main application of the result.

Definition 3.2.3 Let H, L be two k -flat Hopf algebras. We call H and L monoidally Morita-Takeuchi equivalent if there is a k -linear monoidal equivalence ${}^H\mathcal{M} \rightarrow {}^L\mathcal{M}$.

Since the monoidal category structure of the comodule category of a Hopf algebra is one of its main features, it should be clear that monoidal Morita-Takeuchi equivalence is an interesting notion of equivalence between two Hopf algebras, weaker than isomorphy. Theorem 3.2.2 immediately implies:

Corollary 3.2.4 *For two k -flat Hopf algebras H and L , the following are equivalent:*

1. *H and L are monoidally Morita-Takeuchi equivalent.*
2. *There exists an L - H -bi-Galois object.*

3. *There is a k -linear monoidal category equivalence $\mathcal{M}^H \rightarrow \mathcal{M}^L$.*

As a consequence of Corollary 3.2.4 and Proposition 3.1.6, Hopf algebras are monoidally Morita-Takeuchi equivalent if they are cocycle double twists of each other (one should note, though, that it is quite easy to give a direct proof of this fact). Conversely, if H is a finite Hopf algebra over a field k , then every H -Galois object is cleft. Thus every Hopf algebra L which is monoidally Morita-Takeuchi equivalent to H is a cocycle double twist of H .

In many examples constructing bi-Galois objects has proved to be a very practicable way of constructing monoidal equivalences between comodule categories. This is true also in the finite dimensional case over a field. The reason seems to be that it is much easier to construct an associative algebra with nice properties, than to construct a Hopf cocycle (or, worse perhaps, a monoidal category equivalence). I will only very briefly give references for such applications: Nice examples involving the representation categories of finite groups were computed by Masuoka [27]. In [28] Masuoka proves that certain infinite families of non-isomorphic pointed Hopf algebras collapse under monoidal Morita-Takeuchi equivalence. That paper also contains a beautiful general mechanism for constructing Hopf bi-Galois objects for quotient Hopf algebras of a certain type. This was applied further, and more examples of families collapsing under monoidal Morita-Takeuchi equivalence were given, in Daniel Didt's thesis [12]. Bichon [4] gives a class of infinite-dimensional examples that also involve non-cleft extensions.

Now we return to the proof of Theorem 3.2.2. First we state and prove (at least sketchily) the part that is independent of k and any assumptions on the antipode.

Lemma 3.2.5 *Let L and H be k -flat Hopf algebras. Then every k -linear equivalence $\mathcal{F}: {}^H\mathcal{M} \rightarrow {}^L\mathcal{M}$ has the form $\mathcal{F}(V) = A \square_H V$ for some L - H -bi-Galois object A .*

More precisely, every exact k -linear functor $\mathcal{F}: {}^H\mathcal{M} \rightarrow {}^L\mathcal{M}$ commuting with arbitrary colimits has the form $\mathcal{F}(V) = A \square_H V$ for an L - H -bicomodule algebra that is an H -Galois object, and if \mathcal{F} is an equivalence, then A is an L -Galois object.

Proof Let B be a k -flat bialgebra, and $\mathcal{F}: {}^H\mathcal{M} \rightarrow {}^L\mathcal{M}$ an exact functor commuting with colimits. We already know that the composition $\mathcal{F}_0: {}^H\mathcal{M} \rightarrow \mathcal{M}_k$ of \mathcal{F} with the underlying functor has the form $\mathcal{F}_0(V) = A \square_H V$ for an H -Galois object A . It is straightforward to check that \mathcal{F} has the form $\mathcal{F}(V) = A \square_H V$ for a suitable L -comodule algebra structure on A making it an L - H -bicomodule algebra (just take the left L -comodule structure of $A = A \square_H H = \mathcal{F}_0(A)$, and do a few easy calculations). Conversely, every B - H -bicomodule algebra structure on A for some flat bialgebra B lifts \mathcal{F}_0 to a monoidal functor $\mathcal{G}: {}^H\mathcal{M} \rightarrow {}^B\mathcal{M}$. If \mathcal{F} is an equivalence, we can fill in the dashed arrow in the diagram

$$\begin{array}{ccc}
 {}^H\mathcal{M} & \xrightarrow{\mathcal{F}} & {}^L\mathcal{M} \\
 \searrow \mathcal{G} & & \downarrow \mathcal{U} \\
 & & {}^B\mathcal{M} \xrightarrow{\mathcal{U}} \mathcal{M}_k
 \end{array}$$

by a monoidal functor. To see this, simply note that every L -module is by assumption naturally isomorphic to one of the form $A \square_H V$ with $V \in {}^H\mathcal{M}$, and thus it is also a B -module. Now a monoidal functor ${}^L\mathcal{M} \rightarrow {}^B\mathcal{M}$ that commutes with the underlying functors has the form ${}^f\mathcal{M}$ for a unique bialgebra map $f: L \rightarrow B$. We

have shown that L has the universal property characterizing the left Hopf algebra $L(A, H)$. \square

Now what is left of the proof of Theorem 3.2.2 is to provide a converse to Lemma 3.2.5.

In the case that k is a field, we can argue by the general principles of reconstruction theory for quantum groups, which also go back to work of Ulbrich [55]; see e.g. [31]. Assume given an H -Galois object A . The restriction $A \square_H - : {}^H\mathcal{M}_f \rightarrow \mathcal{M}_k$ of the functor $A \square_H -$ to the category of finite-dimensional H -comodules takes values in finite dimensional vector spaces (see Corollary 2.5.5). Thus there exists a Hopf algebra L such that the functor factors over an equivalence ${}^H\mathcal{M}_f \rightarrow {}^L\mathcal{M}_f$; by the finiteness theorem for comodules this also yields an equivalence ${}^H\mathcal{M} \rightarrow {}^L\mathcal{M}$. By Lemma 3.2.5 we see that this equivalence comes from an L - H -bi-Galois structure on A , and in particular that cotensoring with a bi-Galois extension A is an equivalence ${}^H\mathcal{M} \rightarrow {}^{L(A,H)}\mathcal{M}$.

The general technique of reconstruction behind this proof is to find a Hopf algebra from a monoidal functor $\omega: \mathcal{C} \rightarrow \mathcal{M}_k$ by means of a coendomorphism coalgebra construction. More generally, one can construct a cohomomorphism object $\text{cohom}(\omega, \nu)$ for every pair of functors $\omega, \nu: \mathcal{C} \rightarrow \mathcal{M}_k$ taking values in finite dimensional vector spaces. Ulbrich in fact reconstructs a Hopf-Galois object from a monoidal functor ${}^H\mathcal{M}_f \rightarrow \mathcal{M}_k$ by applying this construction to the monoidal functor in question on one hand, and the underlying functor on the other hand. It is clear that the left Hopf algebra of a Hopf-Galois object A can be characterized as the universal Hopf algebra reconstructed as a coendomorphism object from the functor $A \square_H -$. Bichon [3] has taken this further by reconstructing a bi-Galois object, complete with both its Hopf algebras, from a pair of monoidal functors $\omega, \nu: \mathcal{C} \rightarrow \mathcal{M}_k$ taking values in finite dimensional vector spaces. He also gives an axiom system (called a Hopf-Galois system, and extended slightly to be symmetric by Grunspan [19]) characterizing the complete set of data arising in a Bi-Galois situation: An algebra coacted upon by two bialgebras, and in addition another bicomodule algebra playing the role of the inverse bi-Galois extension.

In the case where k is not a field, reconstruction techniques as the ones used above are simply not available, and we have to take a somewhat different approach. If we can show that \underline{BiGal} is a groupoid, then the rest of Theorem 3.2.2 follows: The inverse of the functor $A \square_H - : {}^H\mathcal{M} \rightarrow {}^L\mathcal{M}$ can be constructed as $A^{-1} \square_L : {}^L\mathcal{M} \rightarrow {}^H\mathcal{M}$ when A^{-1} is the inverse of A in the groupoid \underline{BiGal} .

Now let A be an L - H -bi-Galois object. By symmetry it is enough to find a right inverse for A . For this in turn it is enough to find some left H -Galois object B such that $A \square_H B \cong L$ as left L -comodule algebras. For B is an H - R -bi-Galois object for some Hopf algebra R , and $A \square_H B$ is then an L - R -bi-Galois object. But if $A \square_H B \cong L$ as left L -comodule algebra, then $R \cong L$ by the uniqueness of the right Hopf algebra in the bi-Galois extension L . More precisely, there is an automorphism of the Hopf algebra L such that $A \square_H B \cong L^f$, where L^f has the right L -comodule algebra structure induced along f . But then $A \square_H (B^{f^{-1}}) \cong L$ as L -bicomodule algebras.

We already know that $L \cong (A \otimes A)^{\text{co}H}$, a subalgebra of $A \otimes A^{\text{op}}$. From the way the isomorphism was obtained in Remark 3.1.7, it is obviously left L -colinear, with the left L -comodule structure on $(A \otimes A)^{\text{co}H}$ induced by that of the left tensor factor A . Thus it finally remains to find some left H -Galois object B such that

$A \square_H B \cong (A \otimes A)^{\text{co}H}$. If the antipode of H is bijective, we may simply take $B := A^{\text{op}}$, with the left comodule structure $A \ni a \mapsto S^{-1}(a_{(1)}) \otimes a_{(0)}$. If the antipode of H is not bijective, we can take $B := (H \otimes A)^{\text{co}H}$, where the coinvariants are taken with respect to the diagonal comodule structure, the algebra structure is that of a subalgebra of $H \otimes A^{\text{op}}$, and the left H -comodule structure is induced by that of H . By contrast to the case where the antipode is bijective, it is not entirely trivial to verify that B is indeed a left H -Galois object. We refer to [33] for details at this point. However, it is easy to see that the obvious isomorphism $A \square_H B = A \square_H (H \otimes A)^{\text{co}H} \cong (A \square_H H \otimes A)^{\text{co}H} \cong (A \otimes A)^{\text{co}H}$ is a left L -comodule algebra map.

3.3 The structure of Hopf bimodules. Let A be an L - H -bi-Galois object. We have studied already in Section 2.6 the monoidal category ${}_A\mathcal{M}_A^H$ of Hopf bimodules, which allows an underlying functor to the category \mathcal{M}_k which is monoidal. The result of this section is another characterization of the left Hopf algebra L : It is precisely that Hopf algebra for which we obtain a commutative diagram of monoidal functors

$$\begin{array}{ccc} {}_A\mathcal{M}_A^H & \xrightarrow{\sim} & {}_L\mathcal{M} \\ & \searrow^{(-)^{\text{co}H}} & \swarrow \\ & \mathcal{M}_k & \end{array} \quad (3.3.1)$$

in which the top arrow is an equivalence, and the unmarked arrow is the underlying functor.

Theorem 3.3.1 *Let A be an L - H -bi-Galois object. Then a monoidal category equivalence ${}_L\mathcal{M} \rightarrow {}_A\mathcal{M}_A^H$ is defined by sending $V \in {}_L\mathcal{M}$ to $V \otimes A$ with the obvious structure of a Hopf module in \mathcal{M}_A^H , and the additional left A -module structure $x(v \otimes y) = x_{(-1)} \cdot v \otimes x_{(0)}y$. The monoidal functor structure is given by the canonical isomorphism $(V \otimes A) \otimes_A (V \otimes A) \cong V \otimes W \otimes A$.*

Proof We know that every Hopf module in \mathcal{M}_A^H has the form $V \otimes A$ for some k -module V . It remains to verify that left A -module structures on $V \otimes A$ making it a Hopf module in ${}_A\mathcal{M}_A^H$ are classified by left L -module structures on V . A suitable left A -module structure is a colinear right A -module map $\mu: A \otimes V \otimes A \rightarrow V \otimes A$, and such maps are in turn in bijection with colinear maps $\sigma: A \otimes V \rightarrow V \otimes A$. By the general universal property of L , such maps σ are in turn classified by maps $\mu_0: L \otimes V \rightarrow V$ through the formula $\sigma(a \otimes v) = a_{(-1)} \cdot v \otimes a_{(0)}$, with $\mu_0(\ell \otimes v) =: \ell \cdot v$. Now it only remains to verify that μ_0 is an L -module structure if and only if μ , which is now given by $\mu(x \otimes v \otimes y) = x_{(-1)} \cdot v \otimes x_{(0)}y$, is an A -module structure. We compute

$$\begin{aligned} x(y(v \otimes z)) &= x(y_{(-1)} \cdot v \otimes y_{(0)}z) = x_{(-1)} \cdot (y_{(-1)} \cdot v) \otimes x_{(0)}y_{(0)}z \\ (xy)(v \otimes z) &= (xy)_{(-1)} \cdot v \otimes (xy)_{(0)}z = (x_{(-1)}y_{(-1)}) \cdot v \otimes x_{(0)}y_{(0)}z \end{aligned}$$

so that the associativity of μ and μ_0 is equivalent by another application of the universal property of L . We skip unitality.

We have seen that the functor in consideration is well-defined and an equivalence. To check that it is monoidal, we should verify that the canonical isomorphism

$f: V \otimes W \otimes A \rightarrow (V \otimes A) \otimes_A (W \otimes A)$ is a left A -module map for $V, W \in L$. Indeed

$$\begin{aligned} f(x(v \otimes w \otimes y)) &= f(x_{(-1)} \cdot (v \otimes w) \otimes x_{(0)}y) = f(x_{(-2)} \cdot v \otimes x_{(-1)} \cdot w \otimes x_{(0)}y) \\ &= x_{(-2)} \cdot v \otimes 1 \otimes x_{(-1)} \cdot w \otimes x_{(0)}y = x_{(-1)} \cdot v \otimes 1 \otimes x_{(0)}(w \otimes y) \\ &= x_{(-1)} \cdot v \otimes x_{(0)} \otimes (w \otimes y) = x(v \otimes 1) \otimes 1 \otimes y = xf(v \otimes w \otimes y) \end{aligned}$$

for all $x, y \in A$, $v \in V$ and $w \in W$. \square

Corollary 3.3.2 *Let A be an L - H -Bi-Galois object. Then there is a bijection between isomorphism classes of*

1. Pairs (T, f) , where T is an H -comodule algebra, and $f: A \rightarrow T$ is an H -comodule algebra map, and
2. L -module algebras R

It is given by $R := T^{\text{co}H}$, and $T := R\#A := R \otimes A$ with multiplication given by $(r\#x)(s\#y) = r(x_{(-1)} \cdot s)\#x_{(0)}y$.

Note in particular that every T as in (1) is a left faithfully flat H -Galois extension of its coinvariants.

Remark 3.3.3 If A is a faithfully flat H -Galois extension of B , then ${}_A\mathcal{M}_A^H$ is still a monoidal category, and the coinvariants functor is still a monoidal functor to ${}_B\mathcal{M}_B$. It is a natural question whether there is still some L whose modules classify Hopf modules in the same way as we have shown in this section for the case $B = k$, and whether L is still a Hopf algebra in any sense. This was answered in [35] by showing that $L = (A \otimes A)^{\text{co}H}$ still yields a commutative diagram (3.3.1), and that L now has the structure of a \times_B -bialgebra in the sense of Takeuchi [48]. These structures have been studied more recently under the name of quantum groupoids or Hopf algebroids. They have the characteristic property that modules over a \times_B -bialgebra still form a monoidal category, so that it makes sense to say that (3.3.1) will be a commutative diagram of monoidal functors. The \times_B -bialgebra L can step in in some cases where the left Hopf algebra L is useful, but $B \neq k$. Since the axiomatics of \times_B -bialgebras are quite complicated, we will not pursue this matter here.

3.4 Galois correspondence. The origin of bi-Galois theory is the construction in [18] of certain separable extensions of fields that are Hopf-Galois with more than one possibility for the Hopf algebra. The paper [18] also contains information about what may become of the classical Galois correspondence between subfields and subgroups in this case. In particular, there are examples of classically Galois field extensions that are also H -Galois in such a way that the quotient Hopf algebras of H correspond one-to-one to the *normal* intermediate fields, that is, to the intermediate fields that are stable under the coaction of the dual Hopf algebra k^G of the group algebra of the Galois group. Van Oystaeyen and Zhang [56] then constructed what we called $L(A, H)$ above for the case of commutative A (and H), and proved a correspondence between quotients of $L(A, H)$ and H -costable intermediate fields in case A is a field. The general picture was developed in [33, 36]. We will not comment on the proof here, but simply state the results.

Theorem 3.4.1 *Let A be an L - H -Bi-Galois object for k -flat Hopf algebras L, H with bijective antipodes.*

A bijection between

- coideal left ideals $I \subset L$ such that L/I is k -flat and L is a faithfully coflat left (resp. right) L/I -comodule, and
- H -subcomodule algebras $B \subset A$ such that B is k -flat and A is a faithfully flat left (resp. right) B -module

is given as follows: To a coideal left ideal $I \subset L$ we assign the subalgebra $B := {}^{\text{co}A}L/I$. To an H -subcomodule algebra $B \subset A$ we assign the coideal left ideal $I \subset L \cong (A \otimes A)^{\text{co}H}$ such that $L/I \cong (A \otimes_B A)^{\text{co}H}$.

Let $I \subset L$ and $B \subset A$ correspond to each other as above. Then

1. I is a Hopf ideal if and only if B is stable under the Miyashita-Ulbrich action of H on A .
2. I is stable under the left coadjoint coaction of L on itself if and only if B is stable under the coaction of L on A .
3. I is a conormal Hopf ideal if and only if B is stable both under the coaction of L and the Miyashita-Ulbrich action of H on A .

As the special case $A = H$, the result contains the quotient theory of Hopf algebras, that is, the various proper Hopf algebra analogs of the correspondence between normal subgroups and quotient groups of a group. See [50, 26].

3.5 Galois objects over tensor products. Let H_1, H_2 be two Hopf algebras. If both H_i are cocommutative, then $\text{Gal}(H_i)$ are groups under cotensor product, as well as $\text{Gal}(H_1 \otimes H_2)$. If both H_i are also commutative, then we have the subgroups of these three groups consisting of all commutative Galois extensions. If, in particular, we take both H_i to be the duals of group algebras of abelian groups, then $H_1 \otimes H_2$ is the group algebra of the direct sum of those two groups, and the groups of commutative Galois objects are the Harrison groups. It is an old result that the functor ‘‘Harrison group’’ is additive. This means that $\text{Har}(H_1 \otimes H_2) \cong \text{Har}(H_1) \oplus \text{Har}(H_2)$ as (abelian) groups. The same result holds true unchanged if we consider general commutative and cocommutative Hopf algebras. However, the same is not true for the complete $\text{Gal}(-)$ groups. A result of Kreimer [24] states very precisely what is true instead: For two commutative cocommutative finitely generated projective Hopf algebras, we have an isomorphism of abelian groups

$$\text{Gal}(H_1) \oplus \text{Gal}(H_2) \oplus \text{Hopf}(H_2, H_1^*) \rightarrow \text{Gal}(H_1 \otimes H_2),$$

where $\text{Hopf}(H_2, H_1^*)$ denotes the set of all Hopf algebra maps from H_2 to H_1^* , which is a group under convolution because H_2 is cocommutative and H_1^* is commutative. The assumption that both Hopf algebras H_i are commutative is actually not necessary. One can also drop the assumption that they be finitely generated projective, if one replaces the summand $\text{Hopf}(H_2, H_1^*)$ by the group (under convolution) $\text{Pair}(H_2, H_1)$ of all Hopf algebra pairings between H_2 and H_1 ; this does not change anything if H_1 happens to be finitely generated projective.

One cannot, however, get away without the assumption of cocommutativity: First of all, of course, we do not have any groups in the case of general H_i . Secondly, some of the information in the above sequence does survive on the level of pointed sets, but not enough to amount to a complete description of $\text{Gal}(H_1 \otimes H_2)$.

As we will show in this section (based on [39]), Bi-Galois theory can come to the rescue to recover such a complete description. Instead of pairings between the Hopf algebras H_i , one has to take into account pairings between the left Hopf algebras L_i in certain H_i -Galois objects.

Lemma 3.5.1 *Let H_1, H_2 be two k -flat Hopf algebras, and A a right H -comodule algebra for $H = H_1 \otimes H_2$.*

We have $A_1 := A^{\text{co}H_2} \cong A \square_H H_1$ and $A_2 := A^{\text{co}H_1} \cong A \square_H H_2$.

A is an H -Galois object if and only if A_i is an H_i -Galois object for $i = 1, 2$.

If this is the case, then multiplication in A induces an isomorphism $A_1 \# A_2 \rightarrow A$, where the algebra structure of $A_1 \# A_2$ is a smash product as in Corollary 3.3.2 for some L_2 -module structure on A_1 , where $L_2 := L(A_2, H_2)$; the H -comodule structure is the obvious one.

Proof It is straightforward to check that $A^{\text{co}H_i} \cong A \square_H H_j$ for $i \neq j$. We know from Proposition 2.5.7 that A_i are Hopf-Galois objects if A is one.

Now assume that A_i is a faithfully flat H_i -Galois extension of k for $i = 1, 2$. By Corollary 3.3.2 we know that multiplication in A induces an isomorphism $A_1 \# A_2 \rightarrow A$ for a suitable L_2 -module algebra structure on A_1 . We view the Galois map $A \otimes A \rightarrow A \otimes H$ as a map of Hopf modules in $\mathcal{M}_{A_2}^{H_2}$. Its H_2 -coinvariant part is the map $A \otimes A_1 \rightarrow A \otimes H_1$ given by $x \otimes y \mapsto xy_{(0)} \otimes y_{(1)}$, which we know to be a bijection. Thus the canonical map for A is a bijection, and A is faithfully flat since it is the tensor product of A_1 and A_2 . \square

To finish our complete description of $H_1 \otimes H_2$ -Galois objects, we need two more consequences from the universal property of the left Hopf algebra:

Lemma 3.5.2 *Let A be an L - H -bi-Galois object, and let R be an L -module algebra and F -comodule algebra for some k -flat bialgebra F . Then $R \# A$ as in Corollary 3.3.2 is an $F \otimes H$ -comodule algebra if and only if it is an F -comodule algebra, if and only if R is an L - F -dimodule in the sense that $(\ell \cdot r)_{(0)} \otimes (\ell \cdot r)_{(1)} = \ell \cdot r_{(0)} \otimes r_{(1)}$ holds for all $r \in R$ and $\ell \in L$.*

Proof Clearly $R \# A$ is an $F \otimes H$ -comodule algebra if and only if it is an F -comodule algebra, since we already know it to be an H -comodule algebra.

Now (ignoring the unit conditions) $R \# A$ is an F -comodule algebra if and only if

$$\begin{aligned} r_{(0)}(x_{(-1)} \cdot s)_{(0)} \# x_{(0)} y \otimes r_{(1)}(x_{(-1)} \cdot s)_{(1)} \\ r_{(0)}(x_{(-1)} \cdot s_{(0)}) \# x_{(0)} y \otimes r_{(1)} s_{(1)} \end{aligned}$$

agree for all $r, s \in R$ and $x, y \in A$. By the universal property of L , this is the same as requiring

$$r_{(0)}(\ell \cdot s)_{(0)} \otimes r_{(1)}(\ell \cdot s)_{(1)} = r_{(0)}(\ell \cdot s_{(0)}) \otimes r_{(1)} s_{(1)}$$

for all $r, s \in R$ and $\ell \in A$, which in turn is the same as requiring the dimodule condition for R . \square

Lemma 3.5.3 *Let A be an L - H -bi-Galois object, B a k -module, and $\mu: B \otimes A \rightarrow A$ an H -colinear map. Then $\mu = \Phi(\tau)$, that is, $\mu(b \otimes a) = \tau(b \otimes a_{(-1)})a_{(0)}$ for a unique $\tau: B \otimes L \rightarrow A$. Moreover,*

1. *Assume that B is a coalgebra. Then μ is a measuring if and only if $\tau(b \otimes \ell m) = \tau(b_{(1)} \otimes \ell)\tau(b_{(2)} \otimes m)$ and $\tau(b \otimes 1) = \varepsilon(b)$ hold for all $b \in C$ and $\ell, m \in L$.*
2. *Assume that B is an algebra. Then μ is a module structure if and only if $\tau(bc \otimes \ell) = \tau(b \otimes \ell_{(2)})\tau(c \otimes \ell_{(1)})$ and $\tau(1 \otimes \ell) = \varepsilon(\ell)$ hold for all $b, c \in B$ and $\ell \in L$.*

3. Assume that B is a bialgebra. Then μ makes A a B -module algebra if and only if τ is a skew pairing between B and L , in the sense of the following definition:

Definition 3.5.4 Let B and L be two bialgebras. A map $\tau: B \otimes L \rightarrow k$ is called a skew pairing if

$$\begin{aligned}\tau(b \otimes \ell m) &= \tau(b_{(1)} \otimes \ell) \tau(b_{(2)} \otimes m), & \tau(b \otimes 1) &= \varepsilon(b) \\ \tau(bc \otimes \ell) &= \tau(b \otimes \ell_{(2)}) \tau(c \otimes \ell_{(1)}), & \tau(1 \otimes \ell) &= \varepsilon(\ell)\end{aligned}$$

hold for all $b, c \in B$ and $\ell, m \in L$. Note that if B is finitely generated projective, then a skew pairing is the same as a bialgebra morphism $L^{\text{cop}} \rightarrow B^*$

Proof We write $\mu(b \otimes a) = b \cdot a$. We have $\mu = \Phi(\tau)$ for $\tau: B \otimes L \rightarrow k$ as a special case of the universal properties of L .

If B is a coalgebra, then

$$\begin{aligned}b \cdot (xy) &= \tau(b \otimes x_{(-1)} y_{(-1)}) x_{(0)} y_{(0)} \\ (b_{(1)} \cdot x)(b_{(2)} \cdot y) &= \tau(b_{(1)} \otimes x_{(-1)}) \tau(b_{(2)} \otimes y_{(-1)}) x_{(0)} y_{(0)}\end{aligned}$$

are the same for all $b \in B, x, y \in A$ if and only if $\tau(b \otimes \ell m) = \tau(b_{(1)} \otimes \ell) \tau(b_{(2)} \otimes m)$ for all $b \in B, \ell, m \in L$, by the universal property again. We omit treating the unit condition for a measuring

If B is an algebra then

$$\begin{aligned}(bc) \cdot x &= \tau(bc \otimes x_{(-1)}) x_{(0)} \\ b \cdot c \cdot x &= \tau(c \otimes x_{(-1)}) b \cdot x_{(0)} = \tau(c \otimes x_{(-2)}) \tau(b \otimes x_{(-1)}) x_{(0)}\end{aligned}$$

agree for all $b, c \in B, x \in A$ if and only if $\tau(bc \otimes \ell) = \tau(c \otimes \ell_{(1)}) \tau(b \otimes \ell_{(2)})$ holds for all $b, c \in B$ and $\ell \in L$. Again, we omit treating the unit condition for a module structure.

Since a module algebra structure is the same as a measuring that is a module structure, we are done. \square

Now we merely need to put together all the information obtained so far to get the following theorem.

Theorem 3.5.5 Let H_1, H_2 be two k -flat Hopf algebras, and put $H = H_1 \otimes H_2$. The map

$$\pi: \text{Gal}(H_1 \otimes H_2) \rightarrow \text{Gal}(H_1) \times \text{Gal}(H_2); A \mapsto (A \square_H H_1, A \square_H H_2)$$

is surjective. For $A_i \in \text{Gal}(H_i)$ let $L_i := L(A_i, H_i)$. The Hopf algebra automorphism groups of L_i act on the right on the set of all skew pairings between L_1 and L_2 . We have a bijection

$$\text{SPair}(L_1, L_2) / \text{CoInn}(L_1) \times \text{CoInn}(L_2) \rightarrow \pi^{-1}(A_1, A_2),$$

given by assigning to the class of a skew pairing τ the algebra $A_1 \#_{\tau} A_2 := A_1 \otimes A_2$ with multiplication $(r \# x)(s \# y) = r \tau(s_{(-1)} \otimes x_{(-1)}) s_{(0)} \# x_{(0)} y$.

In particular, we have an exact sequence

$$\text{CoInn}(H_1) \times \text{CoInn}(H_2) \rightarrow \text{SPair}(L_2, L_1) \rightarrow \text{Gal}(H_1 \otimes H_2) \rightarrow \text{Gal}(H_1) \times \text{Gal}(H_2)$$

Proof Since $\pi(A_1 \otimes A_2) = (A_1, A_2)$, the map π is onto. Fix $A_i \in \text{Gal}(H_i)$. Then the inverse image of $A_1 \otimes A_2$ under π consists of all those H -Galois objects A for which $A \square_H H_i \cong A_i$. By the discussion preceding the theorem, every such A has the form $A = A_1 \# A_2$, with multiplication given by an L_2 -module algebra structure on A_1 , which makes A_1 an L_2 - H_1 -dimodule, and is thus given by a skew pairing between L_1 and L_2 .

Assume that for two skew pairings τ, σ we have an isomorphism $f: A_1 \#_\tau A_2 \rightarrow A_1 \#_\sigma A_2$. Then f has the form $f = f_1 \otimes f_2$ for automorphisms f_i of the H_i -comodule algebra A_i , which are given by $f_i(x) = u_i(x_{(-1)})x_{(0)}$ for algebra maps $u_i: L_i \rightarrow k$. The map f is an isomorphism of algebras if and only if $f((1\#x)(r\#1)) = f(1\#x)f(r\#1)$ for all $r \in A_1$ and $x \in A_2$. Now

$$\begin{aligned} f((1\#x)(r\#1)) &= \tau(r_{(-1)} \otimes x_{(-1)})f(r_{(0)}\#x_{(0)}) \\ &= \tau(r_{(-2)}u_1(r_{(-1)}) \otimes x_{(-2)}u_2(x_{(-1)}))r_{(0)}\#x_{(0)} \end{aligned}$$

and on the other hand

$$f(1\#x)f(r\#1) = u_1(r_{(-2)})u_2(x_{(-2)})\tau(r_{(-1)} \otimes x_{(-1)})r_{(0)}\#x_{(0)}$$

These two expressions are the same for all r, x if and only if τ and σ agree up to composition with $\text{coinn}(u_1) \otimes \text{coinn}(u_2)$, by yet another application of the universal properties of L_1 and L_2 . \square

3.6 Reduction. We take up once again the topic of reduction of the structure group, or the question of when an H -Galois extension reduces to a Q -Galois extension for a quotient Hopf algebra Q of H . We treated the case of a general base B of the extension in Section 2.7. Here, we treat some aspects that are more or less special to the case of a trivial coinvariant subring k , and involve the left Hopf algebra L .

We start by a simple reformulation of the previous results, using Corollary 3.1.4:

Corollary 3.6.1 *Assume the hypotheses of Theorem 2.7.1. Consider the map $\pi: \text{Gal}(Q) \rightarrow \text{Gal}(H)$ given by $\pi(\bar{A}) = A \square_Q H$, and let $A \in \text{Gal}(H)$. Then $\pi^{-1}(A) \cong \text{Hom}_{-H}^-(K, A) / \text{Alg}(L, k)$, where $L = L(A, H)$.*

The criterion we have given above for reducibility of the structure quantum group (i.e. the question when an H -Galois extension comes from a Q -Galois extension) is ‘‘classical’’ in the sense that analogous results are known for principal fiber bundles: If we take away the Miyashita-Ulbrich action on A^B , which is a purely noncommutative feature, we have to find a colinear algebra map $K \rightarrow A$, which is to say an equivariant map from the principal bundle (the spectrum of A) to the coset space of the structure group under the subgroup we are interested in. Another criterion looks even simpler in the commutative case: According to [11, III §4, 4.6], a principal fiber bundle, described by a Hopf-Galois extension A , can be reduced if and only if the associated bundle with fiber the coset space of the subgroup in the structure group admits a section. In our terminology, this means that there is an algebra map $(A \otimes K)^{\text{co}H} \rightarrow B$ of the obvious map $B \rightarrow A$; alternatively, one may identify the associated bundle with $A^{\text{co}Q} \cong (A \otimes K)^{\text{co}H}$, see below. As it turns out, this criterion can be adapted to the situation of general Hopf-Galois extensions as well. In the noncommutative case, there are, again, extra requirements on the map $A^{\text{co}Q} \rightarrow B$. In fact suitable such conditions were spelled out in [7, Sec.2.5],

although the formulas there seem to defy a conceptual interpretation. As it turns out, the extra conditions can be cast in a very simple form using the left Hopf algebra L : The relevant map $A^{\text{co}Q} \rightarrow B$ should simply be L -linear with respect to the Miyashita-Ulbrich action of L . Since the result now involves the left Hopf algebra, it can only be formulated like that in the case $B = k$; we note, however, that the result, as well as its proof, is still valid for the general case — one only has to take the \times_B -bialgebra L , see Remark 3.3.3, in place of the ordinary bialgebra L .

Theorem 3.6.2 *Assume the hypotheses of Theorem 2.7.1, and let A be an L - H -Bi-Galois object. $A^{\text{co}Q} \subset A$ is a submodule with respect to the Miyashita-Ulbrich action of L on A .*

The following are equivalent:

1. $A \in \text{Gal}(H)$ is in the image of $\text{Gal}(Q) \ni \bar{A} \mapsto \bar{A} \square_Q H \in \text{Gal}(H)$.
2. There is an L -module algebra map $A^{\text{co}Q} \rightarrow k$.

Proof The inverse of the Galois map of the left L -Galois extension A maps L to $(A \otimes A)^{\text{co}H}$, so it is straightforward to check that $A^{\text{co}Q}$ is invariant under the Miyashita-Ulbrich action of L .

We have an isomorphism $\theta_0: A \rightarrow (A \otimes H)^{\text{co}H}$ with $\theta(a) = a_{(0)} \otimes S(a_{(1)})$ and $\theta_0^{-1}(a \otimes h) = a\varepsilon(h)$. One checks that $\theta_0(a) \in (A \otimes K)^{\text{co}H}$ if and only if $a \in A^{\text{co}Q}$, so that we have an isomorphism $\theta: A^{\text{co}Q} \rightarrow (A \otimes K)^{\text{co}H}$ given by $\theta(a) = a_{(0)} \otimes S(a_{(1)})$. It is obvious that θ is an isomorphism of algebras, if we regard $(A \otimes K)^{\text{co}H}$ as a subalgebra of $A \otimes K^{\text{op}}$, but we would like to view this in a more complicated way: Since K is an algebra in the center of \mathcal{M}^H , we can endow $A \otimes K$ with the structure of an algebra in \mathcal{M}^H by setting $(a \otimes x)(b \otimes y) = ab_{(0)} \otimes (x \leftarrow b_{(1)})y$. If $x \otimes a \in (A \otimes K)^{\text{co}H}$, then $(a \otimes x)(b \otimes y) = ab_{(0)} \otimes (x \leftarrow b_{(1)})y = ab_{(0)} \otimes y_{(0)}(x \leftarrow b_{(1)})y_{(1)} = ab \otimes yx$, so $(A \otimes K)^{\text{co}H}$ is a subalgebra of $A \otimes K^{\text{op}}$.

Now the obvious map $A \rightarrow A \otimes K$ is an H -colinear algebra map, so $A \otimes K$ is an algebra in the monoidal category ${}_A\mathcal{M}_A^H$ and hence $(A \otimes K)^{\text{co}H}$ is an L -module algebra by Corollary 3.3.2. Writing $\ell^{(1)} \otimes \ell^{(2)}$ for the preimage of $\ell \otimes 1$ under the Galois map for the left L -Galois extension A , we can compute the relevant L -module structure as $\ell \triangleright (a \otimes x) = \ell^{(1)}(a \otimes x)\ell^{(2)} = \ell^{(1)}a\ell^{(2)}_{(0)} \otimes x \leftarrow \ell^{(2)}_{(1)}$. It is immediate that θ^{-1} is L -linear.

Now we have a bijection between H -colinear maps $f: K \rightarrow A$ and H -colinear and left A -linear maps $\hat{f}: A \otimes K \rightarrow A$ given by $f(x) = \hat{f}(1 \otimes x)$ and $\hat{f}(a \otimes x) = af(x)$. Let us check that f is a right H -module algebra map if and only if \hat{f} is an A -ring map in \mathcal{M}^H . First, assume that f is an H -module algebra map. Then

$$\begin{aligned} \hat{f}((a \otimes x)(b \otimes y)) &= \hat{f}(ab_{(0)} \otimes (x \leftarrow b_{(1)})y) \\ &= ab_{(0)}f(x \leftarrow b_{(1)})f(y) \\ &= ab_{(0)}(f(x) \leftarrow b_{(1)})f(y) \\ &= af(x)bf(y) = \hat{f}(a \otimes x)\hat{f}(b \otimes y), \end{aligned}$$

and $\hat{f}(1 \otimes a) = a$. Conversely, assume that \hat{f} is an A -ring morphism in \mathcal{M}^H . Then f is trivially an algebra map, and

$$f(x)a = \hat{f}(1 \otimes x)\hat{f}(a \otimes 1) = \hat{f}((1 \otimes x)(a \otimes 1)) = \hat{f}(a_{(0)} \otimes x \leftarrow a_{(1)}) = a_{(0)}f(x \leftarrow a_{(1)})$$

for all $a \in A$ and $x \in K$ implies that f is H -linear.

Finally, we already know that A -ring morphisms $\hat{f}: A \otimes K \rightarrow A$ in \mathcal{M}^H , that is, algebra maps in ${}_A\mathcal{M}_A^H$, are in bijection with L -module algebra maps $g: (A \otimes K)^{\text{co}H} \rightarrow k$. \square

If we want to reduce the right Hopf algebra in an L - H -bi-Galois extension, it is of course also a natural question what happens to the left Hopf algebra in the process:

Lemma 3.6.3 *Assume the situation of Theorem 2.7.1. Let \bar{A} be a Q -Galois object, and A the corresponding H -Galois object. Then for any $V \in \mathcal{M}^H$ the map $(V \otimes A)^{\text{co}H} \rightarrow (V \otimes \bar{A})^{\text{co}Q}$ induced by the surjection $A \rightarrow \bar{A}$ is an isomorphism.*

Proof It is enough to check that $\alpha: (V \otimes A)^{\text{co}H} \rightarrow (V \otimes \bar{A})^{\text{co}Q}$ is bijective after tensoring with \bar{A} . We compose $\alpha \otimes \bar{A}$ with the isomorphism $(V \otimes \bar{A})^{\text{co}Q} \otimes \bar{A} \rightarrow V \otimes \bar{A}$ from the structure theorem for Hopf modules in $\mathcal{M}_{\bar{A}}^Q$, and have to check that

$$(V \otimes A)^{\text{co}H} \otimes \bar{A} \rightarrow (V \otimes \bar{A})^{\text{co}Q} \otimes \bar{A} \rightarrow V \otimes \bar{A}; \quad v \otimes a \otimes \bar{b} \mapsto v \otimes a\bar{b}$$

is bijective. But this is the image under the equivalence $\mathcal{M}_K^H \rightarrow \mathcal{M}^Q$ of the isomorphism

$$(V \otimes A)^{\text{co}H} \otimes A \rightarrow V \otimes A; \quad v \otimes a \otimes b \mapsto v \otimes ab$$

from the structure theorem for Hopf modules in \mathcal{M}_A^H . \square

Theorem 3.6.4 *Assume the situation of Theorem 2.7.1, let A be an H -Galois object, $f: K \rightarrow A$ a Yetter-Drinfeld algebra map, and $\bar{A} = A/Af(K^+)$ the corresponding Q -Galois object.*

Using the identification $L := L(A, H) = (A \otimes A)^{\text{co}H}$, the left Hopf algebra of \bar{A} is given by

$$L(\bar{A}, Q) = (A \otimes_K A)^{\text{co}H},$$

where the K -module structure of A is induced via f .

Proof We have to verify that the surjection $A \rightarrow \bar{A}$ induces an isomorphism $(A \otimes_K A)^{\text{co}H} \rightarrow (\bar{A} \otimes \bar{A})^{\text{co}Q}$. Since $(-)^{\text{co}H}: \mathcal{M}_A^H \rightarrow \mathcal{M}_k$ is an equivalence, this amounts to showing that we have a coequalizer

$$(A \otimes K \otimes A)^{\text{co}H} \rightrightarrows (A \otimes A)^{\text{co}H} \rightarrow (\bar{A} \otimes \bar{A})^{\text{co}Q} \rightarrow 0.$$

Using Lemma 3.6.3 this means a coequalizer

$$(A \otimes K \otimes \bar{A})^{\text{co}Q} \rightrightarrows (A \otimes \bar{A})^{\text{co}Q} \rightarrow (\bar{A} \otimes \bar{A})^{\text{co}Q} \rightarrow 0.$$

Since $(-)^{\text{co}Q}: \mathcal{M}_{\bar{A}}^Q \rightarrow \mathcal{M}_k$ is an equivalence, we may consider this before taking the Q -coinvariants, when it is just the definition of $\bar{A} = A/Af(K^+)$ tensored with \bar{A} . \square

4 Appendix: Some tools

4.1 Monoidal category theory. A monoidal category $\mathcal{C} = (\mathcal{C}, \otimes, \Phi, I, \lambda, \rho)$ consists of a category \mathcal{C} , a bifunctor $\otimes: \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$, a natural isomorphism $\Phi: (X \otimes Y) \otimes Z \rightarrow X \otimes (Y \otimes Z)$, an object I , and natural isomorphisms $\lambda: I \otimes X \rightarrow X$ and $\rho: X \otimes I \rightarrow X$, all of which are coherent. This means that all diagrams that one can compose from Φ (which rearranges brackets), λ, ρ (which cancel instances of the unit object I) and their inverses commute. By Mac Lane's coherence theorem, it is actually enough to ask for one pentagon of Φ 's, and one triangle with λ, ρ , and Φ , to commute in order that all diagrams commute. A monoidal category is called strict if Φ, λ , and ρ are identities.

The easiest example of a monoidal category is the category \mathcal{M}_k of modules over a commutative ring, with the tensor product over k and the canonical isomorphisms expressing associativity of tensor products. Similarly, the category ${}_R\mathcal{M}_R$ of bimodules over an arbitrary ring R is monoidal with respect to the tensor product over R . We are interested in monoidal category theory because of its very close connections with Hopf algebra theory. If H is a bialgebra, then both the category of, say, left H -modules, and the category of, say, right H -comodules have natural monoidal category structures. Here, the tensor product of $V, W \in {}_H\mathcal{M}$ (resp. $V, W \in \mathcal{M}^H$) is $V \otimes W$, the tensor product over k , equipped with the diagonal module structure $h(v \otimes w) = h_{(1)}v \otimes h_{(2)}w$ (resp. the codiagonal comodule structure $\delta(v \otimes w) = v_{(0)} \otimes w_{(0)} \otimes v_{(1)}w_{(1)}$). The unit object is the base ring k with the trivial module (resp. comodule) structure induced by the counit ε (resp. the unit element of H). Since the associativity and unit isomorphisms in all of these examples are "trivial", it is tempting never to mention them at all, practically treating all our examples as if they were strict monoidal categories; we will do this in all of the present paper. In fact, this sloppiness is almost justified by the fact that every monoidal category is monoidally equivalent (see below) to a strict one. For the examples in this paper, which are categories whose objects are sets with some algebraic structure, the sloppiness is even more justified [40].

A weak monoidal functor $\mathcal{F} = (\mathcal{F}, \xi, \xi_0): \mathcal{C} \rightarrow \mathcal{D}$ consists of a functor $\mathcal{F}: \mathcal{C} \rightarrow \mathcal{D}$, a natural transformation $\xi: \mathcal{F}(X) \otimes \mathcal{F}(Y) \rightarrow \mathcal{F}(X \otimes Y)$ and a morphism $\xi_0: \mathcal{F}(I) \rightarrow I$ making the diagrams

$$\begin{array}{ccccc} (\mathcal{F}(X) \otimes \mathcal{F}(Y)) \otimes \mathcal{F}(Z) & \xrightarrow{\xi \otimes 1} & \mathcal{F}(X \otimes Y) \otimes \mathcal{F}(Z) & \xrightarrow{\xi} & \mathcal{F}((X \otimes Y) \otimes Z) \\ \downarrow \Phi & & & & \downarrow \mathcal{F}(\Phi) \\ \mathcal{F}(X) \otimes (\mathcal{F}(Y) \otimes \mathcal{F}(Z)) & \xrightarrow{1 \otimes \xi} & \mathcal{F}(X) \otimes \mathcal{F}(Y \otimes Z) & \xrightarrow{\xi} & \mathcal{F}(X \otimes (Y \otimes Z)) \end{array}$$

commute and satisfying

$$\begin{aligned} \mathcal{F}(\lambda)\xi(\xi_0 \otimes id) &= \lambda: I \otimes \mathcal{F}(X) \rightarrow \mathcal{F}(X) \\ \mathcal{F}(\rho)\xi(id \otimes \xi_0) &= \rho: \mathcal{F}(X) \otimes I \rightarrow \mathcal{F}(X). \end{aligned}$$

A standard example arises from a ring homomorphism $R \rightarrow S$. The restriction functor ${}_S\mathcal{M}_S \rightarrow {}_R\mathcal{M}_R$ is a weak monoidal functor, with $\xi: M \otimes_R N \rightarrow M \otimes_S N$ for $M, N \in {}_S\mathcal{M}_S$ the canonical surjection.

A monoidal functor is a weak monoidal functor in which ξ and ξ_0 are isomorphisms. Typical examples are the underlying functors ${}_H\mathcal{M} \rightarrow \mathcal{M}_k$ and $\mathcal{M}^H \rightarrow \mathcal{M}_k$

for a bialgebra H . In this case, the morphisms ξ, ξ_0 are even identities; we shall say that we have a strict monoidal functor.

A prebraiding for a monoidal category \mathcal{C} is a natural transformation $\sigma_{XY}: X \otimes Y \rightarrow Y \otimes X$ satisfying

$$\begin{aligned}\sigma_{X,Y \otimes Z} &= (Y \otimes \sigma_{XZ})(\sigma_{XY} \otimes Z): X \otimes Y \otimes Z \rightarrow Y \otimes Z \otimes X \\ \sigma_{X \otimes Y,Z} &= (\sigma_{XZ} \otimes Y)(X \otimes \sigma_{YZ}): X \otimes Y \otimes Z \rightarrow Z \otimes X \otimes Y \\ \sigma_{XI} &= \sigma_{IX} = id_X\end{aligned}$$

A braiding is a prebraiding that is an isomorphism. A symmetry is a braiding with $\sigma_{XY} = \sigma_{YX}^{-1}$. The notion of a symmetry captures the properties of the monoidal category of modules over a commutative ring. For the topological flavor of the notion of braiding, we refer to Kassel's book [22]. We call a (pre)braided category a category with a (pre)braiding.

The (weak) center construction produces a (pre)braided monoidal category from any monoidal category: Objects of the weak center $\mathcal{Z}_0(\mathcal{C})$ are pairs $(X, \sigma_{X,-})$ in which $X \in \mathcal{C}$, and $\sigma_{XY}: X \otimes Y \rightarrow Y \otimes X$ is a natural transformation satisfying

$$\sigma_{X,Y \otimes Z} = (Y \otimes \sigma_{XZ})(\sigma_{XY} \otimes Z): X \otimes Y \otimes Z \rightarrow Y \otimes Z \otimes X$$

for all $Y, Z \in \mathcal{C}$, and $\sigma_{XI} = id_X$. The weak center is monoidal with tensor product $(X, \sigma_{X,-}) \otimes (Y, \sigma_{Y,-}) = (X \otimes Y, \sigma_{X \otimes Y,-})$, where

$$\sigma_{X \otimes Y,Z} = (\sigma_{XZ} \otimes Y)(X \otimes \sigma_{YZ}): X \otimes Y \otimes Z \rightarrow Z \otimes X \otimes Y$$

for all $Z \in \mathcal{C}$, and with neutral element $(I, \sigma_{I,-})$, where $\sigma_{IZ} = id_Z$. The weak center is prebraided with the morphism σ_{XY} as the prebraiding of X and Y . The center $\mathcal{Z}(\mathcal{C})$ consists of those objects $(X, \sigma_{X,-}) \in \mathcal{Z}_0(\mathcal{C})$ in which all σ_{XY} are isomorphisms.

The main example of a (pre)braided monoidal category which we use in this paper is actually a center. Let H be a Hopf algebra. The category $\mathcal{Z}_0(\mathcal{M}^H)$ is equivalent to the category \mathcal{YD}_H^H of right-right Yetter-Drinfeld modules, whose objects are right H -comodules and right H -modules V satisfying the condition

$$(v \leftarrow h)_{(0)} \otimes (v \leftarrow h)_{(1)} = v_{(0)} \leftarrow h_{(2)} \otimes S(h_{(1)})v_{(1)}h_{(3)}$$

for all $v \in V$ and $h \in H$. A Yetter-Drinfeld module V becomes an object in the weak center by

$$\sigma_{VW}(v \otimes w) = w_{(0)} \otimes v \leftarrow w_{(1)}$$

for all $v \in V$ and $w \in W \in \mathcal{M}^H$. It is an object in the center if and only if H has bijective antipode, in which case $\sigma_{VW}^{-1}(w \otimes v) = v \leftarrow S^{-1}(w_{(1)}) \otimes w_{(0)}$.

4.2 Algebras in monoidal categories. At some points in this paper we have made free use of the notion of an algebra within a monoidal category, modules over it, and similar notions. In this section we will spell out (without the easy proofs) some of the basic facts. It is possible that the notion of center that we define below is new.

Let \mathcal{C} be a monoidal category, which we assume to be strict for simplicity. An algebra in \mathcal{C} is an object A with a multiplication $\nabla: A \otimes A \rightarrow A$ and a unit $\eta: I \rightarrow A$ satisfying associativity and the unit condition that $A \cong A \otimes I \xrightarrow{A \otimes \eta} A \otimes A \xrightarrow{\nabla} A$ (and a symmetric construction) should be the identity. It should be clear what a morphism of algebras is. A left A -module in \mathcal{C} is an object M together with a module structure $\mu: A \otimes M \rightarrow M$ which is associative and fulfills an obvious unit

condition. It is clear how to define right modules and bimodules in a monoidal category.

An algebra in the (pre)braided monoidal category \mathcal{C} is said to be commutative if $\nabla_A = \nabla_A \sigma: A \otimes A \rightarrow A$. Obviously we can say that a subalgebra B in A (or an algebra morphism $\iota: B \rightarrow A$) is central in A if

$$\nabla_A(\iota \otimes A) = \nabla_A \sigma_{AA}(\iota \otimes A) = \nabla_A(A \otimes \iota) \sigma_{BA}: B \otimes A \rightarrow A.$$

Since the last notion needs only the braiding between B and A to be written down, it can be generalized as follows:

Definition 4.2.1 Let A be an algebra in \mathcal{C} . A morphism $f: V \rightarrow A$ in \mathcal{C} from an object $V \in \mathcal{Z}_0(\mathcal{C})$ is called central, if $\nabla_A(f \otimes A) = \nabla_A(A \otimes f) \sigma_{VA}: V \otimes A \rightarrow A$.

A center of A is a couniversal central morphism $c: C \rightarrow A$, that is, an object $C \in \mathcal{Z}_0(\mathcal{C})$ with a morphism $c: C \rightarrow A$ in \mathcal{C} such that every central morphism $f: V \rightarrow A$ factors through a morphism $g: V \rightarrow C$ in $\mathcal{Z}_0(\mathcal{C})$.

It is not clear whether a center of an algebra A exists, or if it does, if it is a subobject in \mathcal{C} , though this is true in our main application Lemma 2.1.9. However, the following assertions are not hard to verify:

Remark 4.2.2 Let A be an algebra in \mathcal{C} , and assume that A has a center (C, c) .

1. Any center of A is isomorphic to C .
2. C is a commutative algebra in $\mathcal{Z}_0(\mathcal{C})$.
3. If R is an algebra in $\mathcal{Z}_0(\mathcal{C})$ and $f: R \rightarrow A$ is central and an algebra morphism in \mathcal{C} , then its factorization $g: R \rightarrow C$ is an algebra morphism.

Let A and B be algebras in the prebraided monoidal category \mathcal{C} . Then $A \otimes B$ is an algebra with multiplication

$$A \otimes B \otimes A \otimes B \xrightarrow{A \otimes \sigma_{BA} \otimes B} A \otimes A \otimes B \otimes B \xrightarrow{\nabla_A \otimes \nabla_B} A \otimes B.$$

Again, this is also true if we merely assume B to be an algebra in the weak center of \mathcal{C} .

If B is a commutative algebra in the weak center of \mathcal{C} , then every right B -module M has a natural left B -module structure

$$B \otimes M \xrightarrow{\sigma_{BM}} M \otimes B \xrightarrow{\mu} M$$

which makes it a B - B -bimodule.

Provided that the category \mathcal{C} has coequalizers, one can define the tensor product of a right A -module M and a left A -module N by a coequalizer

$$M \otimes A \otimes N \rightrightarrows M \otimes N \rightarrow M \otimes_A N.$$

If M is an L - A -bimodule, and N is an A - R -bimodule, then $M \otimes_A N$ is an L - R -bimodule provided that tensoring on the left with L and tensoring on the right with R preserves coequalizers. The extra condition is needed to show, for example, that $L \otimes (M \otimes_A N) \rightarrow M \otimes_A N$ is well-defined, using that $L \otimes (M \otimes_A N) \cong (L \otimes M) \otimes_A N$, which relies on $L \otimes -$ preserving the relevant coequalizer.

Some more technicalities are necessary to assure that the tensor product of three bimodules is associative. Assume given in addition an S - R -bimodule T such

that $T \otimes -$ and $S \otimes -$ preserve coequalizers. Since colimits commute with colimits, $T \otimes_S -$ also preserves coequalizers, and we have in particular a coequalizer

$$T \otimes_S (M \otimes A \otimes N) \rightrightarrows T \otimes_S (M \otimes N) \rightarrow T \otimes_S (M \otimes_A N).$$

To get the desired isomorphism

$$T \otimes_L (M \otimes_A N) \cong (T \otimes_L M) \otimes_A N,$$

we need to compare this to the coequalizer

$$(T \otimes_S M) \otimes A \otimes N \rightrightarrows (T \otimes_S M) \otimes N \rightarrow (T \otimes_S M) \otimes_A N,$$

which can be done if we throw in the extra condition that the natural morphism $(T \otimes_S M) \otimes X \rightarrow T \otimes_S (M \otimes X)$ is an isomorphism for all $X \in \mathcal{C}$.

4.3 Cotensor product. To begin with, the cotensor product of comodules is nothing but a special case of the tensor product of modules in monoidal categories: A k -coalgebra C is an algebra in the opposite of the category of k -modules, so the cotensor product of a right C -comodule M and a left C -comodule N (two modules in the opposite category) is defined by an equalizer

$$0 \rightarrow M \square_C N \rightarrow M \otimes N \rightrightarrows M \otimes C \otimes N.$$

We see that the cotensor product of a B - C -bicomodule M and a C - D -bicomodule N is a B - D -bicomodule provided that B and C are flat k -modules. Since flatness of C is even needed to make sense of equalizers within the category of C -comodules, it is assumed throughout this paper that all coalgebras are flat over k .

A right C -comodule V is called C -coflat if the cotensor product functor $V \square_C - : {}^C\mathcal{M} \rightarrow \mathcal{M}_k$ is exact. Since $V \square_C (C \otimes W) = V \otimes W$ for any k -module W , this implies that V is k -flat. If V is k -flat, it is automatic that $V \square_C -$ is left exact. Also, $V \square_C -$ commutes with (infinite) direct sums. From this we can deduce

Lemma 4.3.1 *If V is a coflat right C -comodule, then for any k -module X and any left C -comodule W the canonical map $(V \square_C W) \otimes X \rightarrow V \square_C (W \otimes X)$ is a bijection.*

In particular, if D is another k -flat coalgebra, W is a C - D -bicomodule, and U is a left D -comodule, then cotensor product is associative:

$$(V \square_C W) \square_D U \cong V \square_C (W \square_D U).$$

Proof The second claim follows from the first and the discussion at the end of the preceding section. For the first, observe first that cotensor product commutes with direct sums, so that the canonical map is bijective with a free module $k^{(I)}$ in place of X . Now we choose a presentation $k^{(I)} \rightarrow k^{(J)} \rightarrow X \rightarrow 0$ of X . Since $V \square_C -$ commutes with this coequalizer, we see that the canonical map for X is also bijective. \square

It is a well-known theorem of Lazard that a module is flat if and only if it is a direct limit of finitely generated projective modules. It is well-known, moreover, that a finitely presented module is flat if and only if it is projective. If k is a field, and C a k -coalgebra, every C -comodule is the direct limit of its finite dimensional subcomodules. Thus the following remarkable characterization of Takeuchi [47, A.2.1] may seem plausible (though of course far from obvious):

Theorem 4.3.2 *Let k be a field, C a k -coalgebra, and V a C -comodule. Then V is coflat if and only if C is injective (that is, an injective object in the category of comodules).*

We refer to [47] for the proof.

In Section 2.5 we have made use of a comodule version of Watts' theorem (which, in the original, states that every right exact functor between module categories is tensor product by a bimodule). For the sake of completeness, we prove the comodule version here:

Lemma 4.3.3 *Let C be a k -flat coalgebra, and $\mathcal{F}: {}^C\mathcal{M} \rightarrow \mathcal{M}_k$ an exact additive functor that commutes with arbitrary direct sums.*

Then there is an isomorphism $\mathcal{F}(M) \cong A \square_C M$, natural in $M \in {}^C\mathcal{M}$, for some comodule $A \in \mathcal{M}^C$ which is k -flat and C -coflat.

Proof We first observe that \mathcal{F} is an \mathcal{M}_k -functor. That is to say, there is an isomorphism $\mathcal{F}(M \otimes V) \cong \mathcal{F}(M) \otimes V$, natural in $V \in \mathcal{M}_k$, which is coherent (which is to say, the two obvious composite isomorphisms $\mathcal{F}(M \otimes V \otimes W) \cong \mathcal{F}(M) \otimes V \otimes W$ coincide, and $\mathcal{F}(M \otimes k) \cong \mathcal{F}(M) \otimes k$ is trivial). We only sketch the argument: To construct $\zeta: \mathcal{F}(M) \otimes V \rightarrow \mathcal{F}(M \otimes V)$, choose a presentation $k^{(I)} \xrightarrow{p} k^{(J)} \rightarrow V$. The map p can be described by a column-finite matrix, which can also be used to define a morphism $\hat{p}: \mathcal{F}(M)^{(I)} \rightarrow \mathcal{F}(M)^{(J)}$, which has both $\mathcal{F}(M) \otimes V$ and (since \mathcal{F} commutes with cokernels) $\mathcal{F}(M \otimes V)$ as its cokernel, whence we get an isomorphism ζ between them. Clearly ζ is natural in M . Naturality in V is proved along with independence of the presentation: Let $k^{(K)} \rightarrow k^{(L)} \rightarrow W$ be a presentation of another k -module W , and $f: V \rightarrow W$. By the Comparison Theorem for projective resolutions, f can be lifted to a pair of maps $f_1: k^{(J)} \rightarrow k^{(L)}$ and $f_2: k^{(I)} \rightarrow k^{(K)}$. Since the maps of free k -modules can be described by matrices, they give rise to a diagram

$$\begin{array}{ccc} \mathcal{F}(M)^{(I)} & \longrightarrow & \mathcal{F}(M)^{(J)} \\ \downarrow & & \downarrow \\ \mathcal{F}(M)^{(K)} & \longrightarrow & \mathcal{F}(M)^{(L)} \end{array}$$

which can be filled to the right both by $\mathcal{F}(M) \otimes f: \mathcal{F}(M) \otimes V \rightarrow \mathcal{F}(M) \otimes W$ and by $\mathcal{F}(M \otimes f): \mathcal{F}(M \otimes V) \rightarrow \mathcal{F}(M \otimes W)$. If $W = V$ and $f = id$, this proves independence of ζ of the resolution, and for a general choice of W and f it proves naturality of ζ . Coherence is now easy to check.

The rest of our claim is now Pareigis' version [30, Thm. 4.2] of Watts' theorem [57]. For completeness, we sketch the proof: Put $A := \mathcal{F}(C)$. Then A is a C -comodule via

$$A = \mathcal{F}(C) \xrightarrow{\mathcal{F}(\Delta)} \mathcal{F}({}^*C \otimes C) \cong \mathcal{F}(C) \otimes C = A \otimes C.$$

The functors \mathcal{F} and $A \square_C -$ are isomorphic, since for $M \in {}^C\mathcal{M}$ we have $M \cong C \square_C M$, that is, we have an equalizer

$$M \rightarrow {}^*C \otimes M \rightrightarrows {}^*C \otimes C \otimes M,$$

which is preserved by \mathcal{F} , and hence yields an equalizer

$$\mathcal{F}(M) \rightarrow A \otimes M \rightrightarrows A \otimes C \otimes M.$$

□

Finally, let us note the following associativity between tensor and cotensor product:

Lemma 4.3.4 *Let C be a coalgebra, A an algebra, M a right A -module, N a left A -module and right C -comodule satisfying the dimodule condition $(am)_{(0)} \otimes (am)_{(1)} = am_{(0)} \otimes m_{(1)}$ for all $a \in M$ and $c \in C$; finally let W be a left C -comodule. There is a canonical map*

$$M \otimes_A (N \square_C V) \rightarrow (M \otimes_A N) \square_C V, \text{ given by } m \otimes (n \otimes v) \mapsto (m \otimes n) \otimes v.$$

If M is flat as A -module, or V is coflat as left C -comodule, then the canonical map is a bijection.

In fact, if M is flat, then $M \otimes_A _$ preserves the equalizer defining the cotensor product. If V is coflat, we may argue similarly using Lemma 4.3.1.

4.4 Convolution and composition. Let C be a k -coalgebra and A a k -algebra. The convolution product

$$f * g = \nabla_A(f \otimes g)\Delta_C: C \rightarrow a$$

defined for any two k -linear maps $f, g: C \rightarrow A$ is ubiquitous in the theory of coalgebras and bialgebras. It makes $\text{Hom}(C, A)$ into an algebra, with the k -dual C^* as a special case. A lemma due to Koppinen (see [29, p.91] establishes a correspondence of convolution with composition (which, of course, is an even more ubiquitous operation throughout all of mathematics):

Lemma 4.4.1 *Let C be a k -coalgebra, and A a k -algebra. Then*

$$T = T_A^C: \text{Hom}(C, A) \ni \varphi \mapsto (a \otimes c \mapsto a\varphi(c_{(1)}) \otimes c_{(2)}) \in \text{End}_{A-}^{-C}(A \otimes C)$$

is an anti-isomorphism of k -algebras, with inverse given by $T^{-1}(f) = (A \otimes \varepsilon_C)f(\eta_A \otimes C)$.

In particular, $\varphi: C \rightarrow A$ is invertible with respect to convolution if and only if $T(\varphi)$ is bijective.

The assertions are straightforward to check. Let us point out that bijectivity of T is a special case of the following Lemma, which contains the facts that $A \otimes V$ is the free A -module over the k -module V , and $W \otimes C$ is the cofree C -comodule over the k -module W :

Lemma 4.4.2 *Let A be a k -algebra, C a k -coalgebra, and V a right C -comodule, and W a left A -module. Then we have an isomorphism*

$$\tilde{T}: \text{Hom}(V, W) \ni \varphi \mapsto (a \otimes v \mapsto a\varphi(v_{(0)}) \otimes v_{(1)}) \in \text{Hom}_{A-}^{-C}(A \otimes V, W \otimes C)$$

4.5 Descent. In this section we very briefly recall the mechanism of faithfully flat descent for extensions of noncommutative rings. This is a very special case of Beck's theorem; a reference is [2].

Definition 4.5.1 Let $\eta: R \subset S$ be a ring extension. A (right) descent data from S to R is a right S -module M together with an S -module homomorphism $\theta: M \rightarrow M \otimes_R S$ (also called a descent data on the module M) making the diagrams

$$\begin{array}{ccc} M & \xrightarrow{\theta} & M \otimes_R S \\ \downarrow \theta & & \downarrow \theta \otimes_R S \\ M \otimes_R S & \xrightarrow{M \otimes_R \eta \otimes_R S} & M \otimes_R S \otimes_R S \end{array} \quad \begin{array}{ccc} M & \xrightarrow{\theta} & M \otimes_R S \\ \parallel & & \downarrow m \\ & & M \end{array}$$

commute (where m is induced by the S -module structure of M). Descent data (M, θ) from S to R form a category $\mathcal{D}(S \downarrow R)$ in an obvious way.

If N is a right R -module, then the induced S -module $N \otimes_R S$ carries a natural descent data, namely the map $\theta: N \otimes_R S \ni n \otimes s \mapsto n \otimes 1 \otimes s \in N \otimes_R S \otimes_R S$. This defines a functor from \mathcal{M}_R to the category of descent data from S to R .

Theorem 4.5.2 (Faithfully flat descent) *Let $\eta: R \subset S$ be an inclusion of rings.*

S is faithfully flat as left R -module if and only if the canonical functor from \mathcal{M}_R to the category of descent data from S to R is an equivalence of categories. If this is the case, the inverse equivalence maps a descent data (M, θ) to

$$M^\theta := \{m \in M \mid \theta(m) = m \otimes 1\}.$$

In particular, for every descent data (M, θ) , the map $f: (M^\theta) \otimes_R S \ni m \otimes s \mapsto ms \in M$ is an isomorphism with inverse induced by θ , i.e. $f^{-1}(m) = \theta(m) \in M^\theta \otimes_R S \subset M \otimes_R S$.

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