$\mathbb{Z}/2\mathbb{Z}$-extensions of pointed fusion categories (joint work with J.-M. Vallin)

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A fusion category $\mathcal{C}$:

- A tensor category with associativity isomorphisms $\alpha(X, Y, Z) : (X \otimes Y) \otimes Z \mapsto X \otimes (Y \otimes Z) \quad (X, Y, Z \in Ob(\mathcal{C}))$ satisfying the **Pentagon condition** :

$$\begin{array}{ccc}
((W \otimes X) \otimes Y) \otimes Z & \xrightarrow{\alpha(W,X,Y) \otimes id_Z} & (W \otimes (X \otimes Y)) \otimes Z \\
\downarrow & & \downarrow \alpha(W, X \otimes Y, Z) \\
\alpha(W \otimes X, Y, Z) & & W \otimes ((X \otimes Y) \otimes Z) \\
\downarrow & & \downarrow id_W \otimes \alpha(X,Y,Z) \\
(W \otimes X) \otimes (Y \otimes Z) & \xrightarrow{\alpha(W,X,Y \otimes Z)} & W \otimes (X \otimes (Y \otimes Z))
\end{array}$$

- A semisimple category with duality $ev_X : X^* \otimes X \mapsto \mathbf{1}$ and $coev_X : \mathbf{1} \mapsto X \otimes X^*$, finitely many (classes of) simple objects $(X_i)_{i=1,..,rk(\mathcal{C})}$ and finite dimensional Hom-spaces :

$$X_i \otimes X_j = \bigoplus_k N_{ij}^k X_k \quad \text{(fusion rule)} \quad \text{and} \quad \mathbf{1} = X_{i_0}.$$

We suppose that the ground field is $\mathbb{C}$ and $X = \mathbf{1} \otimes X = X \otimes \mathbf{1}$.

Dimensions

Frobenius-Perron dimension of X_i - the largest nonnegative eigenvalue of N_{ij}^k. We have

$$FPdim(X_i \otimes X_j) = FPdim(X_i)FPdim(X_j),$$

$$FPdim(X_i \oplus X_j) = FPdim(X_i) + FPdim(X_j)$$

which gives a homomorphism of the **fusion ring** of $\mathcal{C}$ to $\mathbb{R}$. By definition, $FPdim(\mathcal{C}) = \Sigma_i FPdim(X_i)^2$.

Proposition ([ENO1], 2005) If $FPdim(\mathcal{C}) \in \mathbb{N}$, then :
1) $\mathcal{C}$ admits a unique **pivotal structure** (i.e., a family of isomorphisms $a_X : X \mapsto X^{**}$ such that $a_{X \otimes Y} = a_X \otimes a_Y$) satisfying $Tr(a_X) = FPdim(X)$, where $Tr(a_X) := ev_{X^*} \circ \circ (a_X \otimes id_{X^*}) \circ coev_X \in End(\mathbf{1}) \cong \mathbb{C}$, $X, Y \in Ob(\mathcal{C})$. Such categories are called **pseudo-unitary**, they are automatically **spherical**, i.e., $Tr(a_X) = Tr(a_{X^*})$.
2) $Tr(a_{X_i}) = FPdim(X_i) = \sqrt{N}_i$, where $N_i \in \mathbb{N}$.

Examples of fusion categories

1) The category of finite dimensional vector spaces (rank 1 fusion category), representation categories of finite groups or finite dimensional semisimple Hopf algebras.

2) $\mathcal{C} = Vec_G^{\omega}$: simple objects are $g, h, k \in G$, fusion rule : $g \otimes h = gh$, duality : $g^* = g^{-1}$, $\alpha(g, h, k) = \omega(g, h, k) Id_{ghk}$, where ω is a 3-cocycle on a finite group G - **Pointed fusion categories**.

3) Categories of bimodules coming from the theory of Von Neumann subfactors of finite index and finite depth.

In particular, Yang-Lee fusion category :
$Ob(\mathcal{C}) = \{\mathbf{1}, X\}$, $X \otimes X = \mathbf{1} \oplus X$,
so $(FPdim(X))^2 = 1 + FPdim(X) \Longrightarrow FPdim(X) = \frac{1+\sqrt{5}}{2}$.

Graded fusion categories :

$$\mathcal{C} = \oplus_{\gamma\in\Gamma}\mathcal{C}_\gamma, \quad \mathcal{C}_a \otimes \mathcal{C}_b \subset \mathcal{C}_{ab}, \ a, b \in \Gamma \quad (\Gamma \text{ is a finite group}).$$

We want to classify $\mathbb{Z}/2\mathbb{Z}$-extensions $\mathcal{C} = \mathcal{C}_0 \oplus \mathcal{C}_1$ of $\mathcal{C}_0 = Vec_G^\omega$

Proposition If $\mathcal{C}_0 = Vec_G^\omega$, then :

- G acts transitively on both sides on the set $Irr(\mathcal{C}_1) = G/A$ of simple objects of $\mathcal{C}_1$, these actions commute and the stabilizer of any simple object is $A \triangleleft G$.
- Fusion rules and duality : for $g, h \in G, M, N \in G/A$, $g \otimes M = g \cdot M$, $M \otimes g = M \cdot g$, $(g \cdot M)^* = M^* \cdot g^{-1}$, $M \otimes N = \bigoplus_{M = g\cdot N^*} g$, and $\{g \in G | M = gN^*\}$ - an A-coset.

Corollary $FPdim(g) = 1, \ \forall g \in G$, $FPdim(M) = \sqrt{|A|}$, $\forall M \in G/A$, so $FPdim(\mathcal{C}) = 2|G|$ and $\mathcal{C}$ is pseudo-unitary.

Example : Tambara-Yamagami categories ([TY], 1998), where $Irr(\mathcal{C}_1) = \{M = M^*\}$. Then $A = G$ must be abelian and $\omega = 1$. They are classified by triples (A, χ, τ), where $\chi : A \times A \to \mathbb{C}^\times$ is a symmetric non-degenerate bicharacter on A and $\tau = \pm|A|^{-1/2}$.

The structure of graded fusion categories ([ENO2], 2010)

Right $\mathcal{C}$-module category $(\mathcal{M}, \tilde{\mu}^r)$: a bifunctor $\odot : \mathcal{M} \times \mathcal{C} \longrightarrow \mathcal{M}$ equipped with associativity isomorphisms $\tilde{\mu}^r : M \odot (X \otimes Y) \mapsto$ $\mapsto (M \odot X) \odot Y$ satisfying the **Pentagon conditions** :

$$\begin{array}{ccc}
M \odot ((X \otimes Y) \otimes Z)) & \xrightarrow{\tilde{\mu}^r(M, X\otimes Y, Z)} & (M \odot (X \otimes Y)) \odot Z \\
\Big\downarrow {\scriptstyle id_M \odot \alpha(X,Y,Z)} & & \Big\downarrow \\
M \odot (X \otimes (Y \otimes Z)) & & \tilde{\mu}^r(M, X, Y) \odot id_Z \\
\Big\downarrow {\scriptstyle \tilde{\mu}^r(M, X, Y\otimes Z)} & & \Big\downarrow \\
(M \odot X) \odot (Y \otimes Z) & \xrightarrow{\tilde{\mu}^r(M\odot X, Y, Z)} & ((M \odot X) \odot Y) \odot Z
\end{array}$$

Right module functor $(F, \gamma) : (\mathcal{M}_1, \tilde{\mu}^r_1) \to (\mathcal{M}_2, \tilde{\mu}^r_2)$ is a functor $F : \mathcal{M}_1 \to \mathcal{M}_2$ equipped with a family of isomorphisms $\gamma(M, X) :$ $F(M \odot X) \to F(M) \odot X$ satisfying

$$
\begin{array}{ccc}
F(M \odot ((X \otimes Y)) & \xrightarrow{\gamma(M, X \otimes Y)} & F(M) \odot (X \otimes Y)) \\
\Big\downarrow{\scriptstyle F(\tilde{\mu}_1^r(M,X,Y))} & & \Big\downarrow \\
F((M \odot X) \odot Y) & & \tilde{\mu}_2^r(F(M), X, Y) \\
\Big\downarrow{\scriptstyle \gamma(M \odot X, Y)} & & \Big\downarrow \\
F(M \odot X) \odot Y & \xrightarrow{\gamma(M,X) \odot id_Y} & (F(M) \odot X) \odot Y
\end{array}
$$

Two right module functors, (F^1, γ^1) and (F^2, γ^2), are said to be isomorphic if there is a family of isomorphisms $\eta(M) : F^1(M) \mapsto F^2(M)$ such that

$$
\begin{array}{ccc}
F^1(M \odot X) & \xrightarrow{\eta(M \odot X)} & F^2(M \odot X) \\
\Big\downarrow{\scriptstyle \gamma^1(M,X)} & & \Big\downarrow{\scriptstyle \gamma^2(M,X)} \\
F^1(M) \odot X & \xrightarrow{\eta(M) \odot id_X} & F^2(M) \odot X
\end{array}
$$

Bimodule categories

A $(\mathcal{C}, \mathcal{D})$-bimodule category is a module category over $\mathcal{C} \boxtimes \mathcal{D}^{op}$, where $\mathcal{D}^{op}$ is the opposite fusion category to $\mathcal{D}$ (i.e., with reversed order of tensor product and inverted associativity isomorphisms) and $\boxtimes$ is Deligne's tensor product of finite abelian linear categories.

Alternatively, a $(\mathcal{C}, \mathcal{D})$-bimodule category, is defined by three structures : right $\mathcal{D}$-module category as above, a left $\mathcal{C}$-module category defined by a bifunctor $\odot : \mathcal{C} \times \mathcal{M} \longrightarrow \mathcal{M}$ equipped with associativity isomorphisms $\tilde{\mu}^l : (X \otimes Y) \odot M \mapsto X \otimes (Y \odot M)$, and also left-right compatibility isomorphisms :

$$\tilde{\chi}(X, M, Y) : (X \odot M) \odot Y \mapsto X \odot (M \odot Y)$$

satisfying the corresponding pentagon conditions ([Greenough], 2010).

Then one can define bimodule functors and their isomorphisms.

Tensor product of module categories over $\mathcal{C}$

Let $(\mathcal{M}, \tilde{\mu}^r)$ and $(\mathcal{N}, \tilde{\mu}^l)$ be right and left $\mathcal{C}$-module categories and $\mathcal{A}$ be an abelian category. A bifunctor $F : \mathcal{M} \times \mathcal{N} \to \mathcal{A}$ is $\mathcal{C}$-balanced if there is a family of isomorphisms

$$b_{M,X,N} : F(M \odot X, N) \to F(M, X \odot N)$$

satisfying some pentagon condition with respect to $\tilde{\mu}^r$ and $\tilde{\mu}^l$.

Tensor product $\mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N}$ is an abelian category equipped with a $\mathcal{C}$-balanced bifunctor $B(\mathcal{M}, \mathcal{N}) : \mathcal{M} \times \mathcal{N} \to \mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N}$ satisfying the universal property : for any $\mathcal{C}$-balanced bifunctor $F : \mathcal{M} \times \mathcal{N} \to \mathcal{A}$ there is a unique functor $F' : \mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N} \to \mathcal{A}$ such that $F = F' \circ B(\mathcal{M}, \mathcal{N})$.

Theorem $\mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N} \cong Fun_{\mathcal{C}}(\mathcal{M}^{op}, \mathcal{N})$ - the category of left $\mathcal{C}$-module functors, where $\mathcal{M}^{op}$ is the opposite (left) $\mathcal{C}$-module category to $\mathcal{M}$ with $X \odot_{op} M = M \odot X^*$ and $\tilde{\mu}$ inverted. Moreover, if $\mathcal{M}$ and $\mathcal{N}$ are $(\mathcal{C}, \mathcal{D})$- and $(\mathcal{D}, \mathcal{E})$-bimodule categories, respectively, then $\mathcal{M} \boxtimes_{\mathcal{D}} \mathcal{N}$ is a $(\mathcal{C}, \mathcal{E})$-bimodule category.

A $(\mathcal{C},\mathcal{D})$-bimodule category $\mathcal{M}$ is invertible if $\mathcal{M}^{op}\boxtimes_{\mathcal{C}}\mathcal{M}\cong\mathcal{D}$, $\mathcal{M}\boxtimes_{\mathcal{D}}\mathcal{M}^{op}\cong\mathcal{C}$ ($\mathcal{M}^{op}$ is the opposite $(\mathcal{D},\mathcal{C})$-bimodule category). Brauer-Picard group of $\mathcal{C}$:= {Classes of invertible $\mathcal{C}$-bimodule categories with product $\boxtimes_{\mathcal{C}}$ and unit $\mathcal{C}$} is finite.

Theorem ([ENO2], 2010) If $\mathcal{C}=\oplus_{\gamma\in\Gamma}\mathcal{C}_\gamma$, then :
1) Each $\mathcal{C}_\gamma$ is an invertible $\mathcal{C}_e$-bimodule category.
2) For all $a,b\in\Gamma$, the tensor product of $\mathcal{C}$ restricts to a $\mathcal{C}_e$-balanced bifunctor $\otimes:\mathcal{C}_a\times\mathcal{C}_b\cong C_{ab}$ which gives rise to a $\mathcal{C}_e$-bimodule equivalence $M_{a,b}:\mathcal{C}_a\boxtimes_{\mathcal{C}_e}\mathcal{C}_b\cong C_{ab}$ such that the $\mathcal{C}_e$-bimodule functors $F_{a,b,c}:=M_{a,bc}(Id_{\mathcal{C}_a}\boxtimes_{\mathcal{C}_e}M_{b,c})$ and $G_{a,b,c}:=M_{ab,c}(M_{a,b}\boxtimes_{\mathcal{C}_e}Id_{\mathcal{C}_c})$ are isomorphic.
3) For all $a,b,c\in\Gamma$, isomorphisms $\alpha_{a,b,c}$ of the above functors viewed as $\mathcal{C}_e$-bimodule functors $(\mathcal{C}_a\boxtimes_{\mathcal{C}_e}\mathcal{C}_b)\boxtimes_{\mathcal{C}_e}\mathcal{C}_c\to$
$\to\mathcal{C}_a\boxtimes_{\mathcal{C}_e}(\mathcal{C}_b\boxtimes_{\mathcal{C}_e}\mathcal{C}_c)$ satisfy some pentagon conditions.

Vice versa, given a homomorphism $c:\Gamma\to BrPic(\mathcal{C}_e):\gamma\mapsto\mathcal{C}_\gamma$ and a collection of equivalences $M_{a,b}:\mathcal{C}_a\boxtimes_{\mathcal{C}_e}\mathcal{C}_b\cong C_{ab}$, one can, if some cohomological obstructions vanish, construct a Γ-extension of $\mathcal{C}_e$ with tensor product $\boxtimes_{\mathcal{C}_e}$ and associativity isomorphisms $\alpha_{a,b,c}$.

Classification of module and bimodule categories over Vec_G^ω

Left Vec_G^ω-module categories are of the form $\mathcal{M}(L,\mu)$, where $L < G$ such that $\omega|_{L\times L\times L} = 1$ in $H^3(L,\mathbb{C}^\times)$ and $\mu \in C^2(L,\mathbb{C}^\times)$ satisfies $\partial^2\mu = \omega|_{L\times L\times L}$ ([Ostrik], 2006). Then $Irr(\mathcal{M}(L,\mu)) = = G/L$ and the induced 2-cochain $\tilde{\mu} \in C^2(G, Fun(G/L,\mathbb{C}^\times))$ defines the associativities. Similarly - right module categories.

Remark If $\omega = 1$, then $\tilde{\mu}^r(M,\cdot,\cdot) \mapsto \tilde{\mu}^r(\mathbf{1},\cdot,\cdot)|_{L\times L} := \mu^r(\cdot,\cdot)$ $(\mathbf{1} = L)$ defines, due to **Shapiro's lemma**, an isomorphism

$$H^n(G,\mathbb{F}un(G/L,\mathbb{C}^\times)) \cong H^n(L,\mathbb{C}^\times).$$

The associativity isomorphisms of $\mathcal{M}(L,\mu)^{op}$ are defined by the 2-cochain induced from $\mu^{op}(s,t) := \mu^{-1}(t^{-1},s^{-1})$.

Then bimodule categories over Vec_G^ω are classified by pairs (L,μ), where $L < G \times G^{op}$ and $\mu \in C^2(L,\mathbb{C}^\times)$ satisfies $\partial^2\mu = (\omega \otimes \omega^{op})|_{L\times L\times L}$. Here

$$\omega^{op}(s^{op},t^{op},u^{op}) := \omega^{-1}(s^{-1},t^{-1},u^{-1}).$$

Cohomology related to $\mathbb{Z}/2\mathbb{Z}$-extensions of Vec_G^{ω} :

Alternatively, let $A_1 < G,\ A_2 < G^{op}$ be such that $L \cap (G \times \{e\}) =$ $= A_1 \times \{e\}$ and $L \cap (\{e\} \times G^{op}) = \{e\} \times A_2$. One can identify $(G \times G^{op})/L$ with G/A_1 and G^{op}/A_2 and show, putting $\mu^l := \mu|_{(A_1 \times \{e\}, A_1 \times \{e\})}$, $\mu^r := \mu|_{(\{e\} \times A_2, \{e\} \times A_2)}$, $\chi := \mu|_{(A_1 \times \{e\}, \{e\} \times A_2)}$, that for all $(s_1, s_2^{op}), (t_1, t_2^{op}) \in G \times G^{op}, M \in (G \times G^{op})/L$:

$$\tilde{\mu}((s_1, s_2^{op}), (t_1, t_2^{op}), M) =$$

$$= \tilde{\chi}(s_1, t_1 \cdot M, t_2)\mu^l(s_1, t_1, M)\mu^r((s_1 t_1) \cdot M, t_2, s_2).$$

The pentagon conditions for the C_0-bimodule category structure $\tilde{\mu}$ give the following cohomological equations :

$$\tilde{\mu}^r(M \cdot s, t, u))\tilde{\mu}^r(M, s, tu)\omega(s, t, u) = \tilde{\mu}^r(M, s, t)\tilde{\mu}^r(M, st, u),$$

$$\tilde{\mu}^r(M, t, u)\tilde{\chi}(s, M, tu) = \tilde{\chi}(s, M \cdot t, u)\tilde{\chi}(s, M, t)\tilde{\mu}^r(s \cdot M, t, u),$$

and similar equations connecting the 2-cochains $\tilde{\mu}^l$ and $\tilde{\chi}$.

An invertible $\mathcal{C}_0$-bimodule structure on $\mathcal{C}_1$

As $\Gamma = \mathbb{Z}/2\mathbb{Z}$, $\mathcal{C}_1$ is invertible and $\mathcal{C}_1 \cong \mathcal{C}_1^{op}$, so $Fun_{\mathcal{C}_0}(\mathcal{C}_1, \mathcal{C}_1) \cong$ $\cong C_0$ if $\mathcal{C}_1$ is viewed as a right $\mathcal{C}_0$-module category (equivalent to $\mathcal{M}(A_2, \mu^r)$). As $\mathcal{C}_0$ is pointed, this is possible iff $A_2 \triangleleft G$ and is abelian ([Naidu], 2007). Similarly for A_1. One can also show that $\chi : A_1 \times A_2 \to \mathbb{C}^\times$ must be a non-degenerate bicharacter.

The equivalence $\mathcal{C}_1 \cong \mathcal{C}_1^{op}$ implies : $A_1 = A_2 = A$, $\mu^r(a, b) \cong \mu^l(b^{-1}, a^{-1})^{-1}$ and also that the bicharacter χ is symmetric. As $A \triangleleft G$ is abelian, we have $G \cong A \rtimes_\rho (G/A)$ - a twisted semidirect product, where G/A acts on A and $\rho \in Z^2(G/A, A)$.

Example : $\mathbb{Z}/4\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z} \rtimes_\rho \mathbb{Z}/2\mathbb{Z}$ (action is trivial, ρ is nontrivial).

We also show that $L \cong (A \times A) \rtimes_{\tilde{\rho}} (G/A)$, where G/A acts on $A \times A$ by $t \cdot (a, b) = (t \cdot a, \varepsilon(t) \cdot b)$, $\varepsilon \in Aut(G/A)$ such that $\varepsilon^2 = Ad(\delta), \varepsilon(\delta) = \delta$ for some $\delta \in S/A$, and $\tilde{\rho} \in Z^2(G/A, A \times A)$ coming from ρ.

Next, we calculate explicitly the equivalences $M_{a,b}$ and the functors $F_{a,b,c}$, $G_{a,b,c}$ as above, for all $a, b, c \in \mathbb{Z}/2\mathbb{Z}$, in terms of the triple $(\omega, \tilde{\mu}^r, \tilde{\chi})$, and show that these functors are isomorphic. This means that the cohomological obstruction $O_3(c)$ (see [ENO2], 2010) vanishes. So we are able to equip $\mathcal{C} = \mathcal{C}_0 \oplus \mathcal{C}_1$ with a quasi-tensor product, i.e., with a bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ such that $\otimes \circ (\otimes \times id_{\mathcal{C}}) \cong \otimes \circ (id_{\mathcal{C}} \times \otimes)$.

Let $\alpha_{a,b,c}$ be isomorphisms of the functors $F_{a,b,c}$ and $G_{a,b,c}$. According to [ENO2], (2010), there is a choice of $\alpha_{a,b,c}$ satisfying the pentagon conditions iff some cohomological obstruction $0_4(c, M) \in H^4(\mathbb{Z}/2\mathbb{Z}, \mathbb{C}^\times)$ vanishes. As $H^4(\mathbb{Z}/2\mathbb{Z}, \mathbb{C}^\times) = \{0\}$, such a choice exists. Moreover, the classes of equivalence of such $\alpha_{a,b,c}$ are described by $H^3(\mathbb{Z}/2\mathbb{Z}, \mathbb{C}^\times) = \mathbb{Z}/2\mathbb{Z}$, so there exactly 2 of them, and one can choose representatives that differ only by the sign of $\alpha_{1,1,1}$. In fact, we are able to calculate $\alpha_{a,b,c}$ explicitly.

This analysis allows to prove :

A characterization of $\mathbb{Z}/2\mathbb{Z}$-extensions of Vec_G^ω

Theorem ([VV], 2012)
(case $G = A \rtimes (G/A), \omega = 1$) :
$\mathbb{Z}/2\mathbb{Z}$-extensions of Vec_G^1 are parameterized by collections $(A, \chi, \tau, \varepsilon, \delta, \psi, \nu)$, where :

- A is a normal abelian subgroup of G,
- χ is a symmetric non-degenerate bicharacter on A,
- $\tau = \pm|A|^{-1/2}$,
- ε is an isomorphism of G/A,
- $\delta \in G/A$ is such that $\varepsilon^2 = Ad(\delta), \varepsilon(\delta) = \delta$,
- $\psi \in Z^1(G/A, Fun(A \times A, \mathbb{C}^\times))$ such that $\frac{\chi}{{}^t\chi} = \partial^1\psi$ and $\psi^{-1}(t, a, b) \cong \psi(\varepsilon(t), b^{-1}, a^{-1})$,
- $\nu \in Z^2(G/A, \mathbb{C}^\times)$ such that $\nu^{-1} \cong \nu \circ (\varepsilon \times \varepsilon)$.

Then $Irr(c(0)) = G, Irr(c(1)) = G/A$ and the fusion rule is :
$x^* = x^{-1}, \quad M^* = \varepsilon^{-1}(M^{-1})\delta, \quad x \otimes y = xy,$
$x \otimes M = x \cdot M, \quad M \otimes x = M\varepsilon(p(x)), \quad M \otimes N = \bigoplus_{M = xN^*} x,$
here $p(x)$ is an A-coset containing x.

The associativity isomorphisms of $\mathbb{Z}/2\mathbb{Z}$-extensions of Vec_G^ω

One can express $\alpha(X, Y, Z)$ for $\mathcal{C} = \mathcal{C}_0 \oplus \mathcal{C}_1$ in terms of $\tilde{\chi}$:

- $\alpha_{0,0,0}(x, y, z) = \omega(x, y, z) id_{xyz}$
- $\alpha_{1,0,0}(K, x, y) = id_{Kxy}$
- $\alpha_{0,1,0}(x, K, y) = \tilde{\chi}(x, K, y) id_{xKy}$
- $\alpha_{0,0,1}(x, y, K) = id_{xyK}$
- $\alpha_{0,1,1}(x, K, L) = \oplus_{K=sL^*} id_{xs}$
- $\alpha_{1,1,0}(K, L, x) = \oplus_{K=sL^*} id_{sx}$
- $\alpha_{1,0,1}(K, x, L) = \oplus_{Kx=sL^*} \tilde{\chi}(s, (xL)^*, x) id_s$
- $\alpha_{1,1,1}(K, L, M)$= the matrix $(\tau \tilde{\chi}^{-1}(s, L^*, t) id_{sM})_{K=sL^*;L=tM^*}$

Remarks. 1. The Tambara-Yamagami case : $A = G$.
2. More complicated description of $\mathbb{Z}/2\mathbb{Z}$-extensions of Vec_G^ω was obtained by J. Liptrap (2010)

Examples

1. If G is abelian, $|G| = 2p$ (p is prime), $A < G$ is non-trivial and such that $\omega|_{A \times A \times A} = 1$.
Analyzing symmetric non-degenerate bicharacters and 2-cocycles on A in various special cases, we have :

Proposition.

(i) If p=2 and $G = \mathbb{Z}/4\mathbb{Z}$, $A = \mathbb{Z}/2\mathbb{Z}$, there are 2 fusion rules and 4 non equivalent fusion categories for each of them (in part, this result was obtained earlier by P. Bonderson, 2007).

(ii) If p=2 and $G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, there are 16 non equivalent fusion categories for any of 3 non-trivial subgroups of G.

(iii) If p is odd prime, so $G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$, then :

a) if $A = (0, \mathbb{Z}/p\mathbb{Z})$, there are 8 non equivalent fusion categories ;

b) if $A = (\mathbb{Z}/2\mathbb{Z}, 0)$, there are 6 non equivalent fusion categories.

Examples and applications

2. Alternating group $G = A_4 \cong A \rtimes \mathbb{Z}/3\mathbb{Z},\ A = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.
There are 8 non equivalent fusion categories.

3. Dihedral group $G = D_p := \mathbb{Z}/p\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$ (p is odd prime).
$A = (\mathbb{Z}/p\mathbb{Z}, 0)$. There are 8 non equivalent fusion categories.

Application to the subfactor theory will be discussed by J.-M. Vallin.

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