Finite C^*-Quantum groupoids and subfactors coming from fusion categories

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(iii) it exists $S : A \longrightarrow A$ a bialgebra anti-isomorphism s. t. :
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If A is a C^*-algebra s.t. $\Delta(a^*) = \Delta(a)^*$: C^*-**Quantum Groupoid**

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A is **regular** if $S^2_{|A_t} = 1$.

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Galois correspondence : $M^G \subset K \subset M \Leftrightarrow$ subgroups of G.

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K is a factor iff I is **connected**.

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- $Bimod_{M_0-M_0}$ with tensor product $\otimes_{M_0}$
- $Rep(B^*)$: finite rank B^*-modules with tensor product :
  $V \boxtimes W = \Delta(1)(V \otimes W)$, $(b \cdot (v \boxtimes w) = b_{(1)} \cdot v \boxtimes b_{(2)} \cdot w)$

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Our aim is to apply this to $\mathbb{Z}/2\mathbb{Z}$- extensions of pointed fusion categories in order to construct and analyze the subfactors associated with them

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- χ non deg. bichar. on A, $\mu^r \in C^2(A, \mathbb{C}^X)$, $\psi \in Z^1(S/A, Fun(A \times A, \mathbb{C}^{\times}))$, $\nu \in C^2(S/A, \mathbb{C}^{\times})$ s.t it exists adeq.inductions :
  $\tilde{\chi} : S \times S/A \times S \to \mathbb{C}^X$, $\tilde{\mu}^r : S/A \times S \times S \to \mathbb{C}^X$

Example $A = S$: **Tambara-Yamagami fusion categories**

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- $\alpha(K, L, M)$= the matrix $(\tau \tilde{\chi}^{-1}(s, L^*, t) id_{sM})_{K=sL^*; L=tM^*}$

The Structure of B

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$\mathbb{Z}/2\mathbb{Z}$-graded extensions of Vec_S^{ω} are spherical

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$dim_a(s) = 1$, $dim_a(M) = \sqrt{|A|}$

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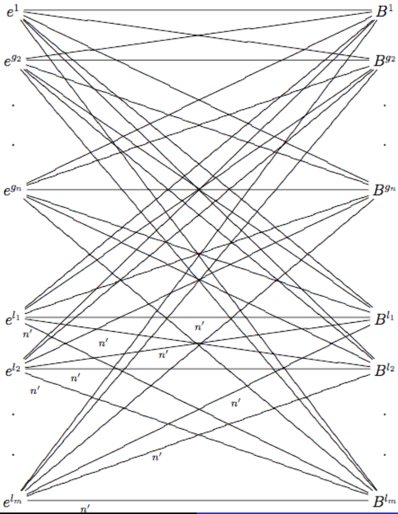
iii) One has : $[B : B_t] = (n + \sqrt{nm})^2 = \gamma^2$.


e^1
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e^{g_n}
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Subfactors attached to Tambara-Yamagami fusion categories

Here the structure of quantum groupoids and description of the tower of type II_1 subfactors associated with them, have already been done

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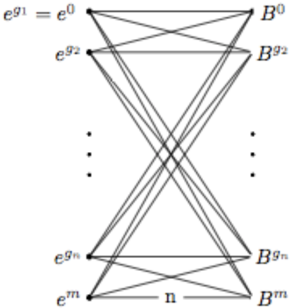
Theorem C. Mevel (10) T.Y. fusion categories give, for all $n \in \mathbb{N}^*$, a subfactors of the hyperfinite type II_1 factor whose index is $(n + \sqrt{n})^2$, and principal graphs is given by :



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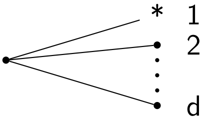

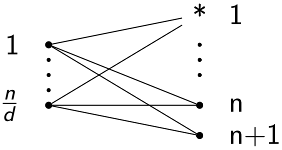

Case 1 : index $= d$ *Case 2 : index* $= \frac{1}{d}(n + \sqrt{n})^2$

Intermediate subfactors

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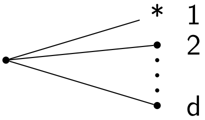

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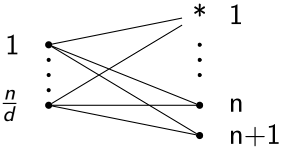

Case 2 : index $= \frac{1}{d}(n + \sqrt{n})^2$

If n is "quadratfrei" these are exactly all possible principal graphs.

The lattice of intermed. subfactors for $S = \mathbb{Z}/p\mathbb{Z}$ and $S = \mathbb{Z}/pq\mathbb{Z}$

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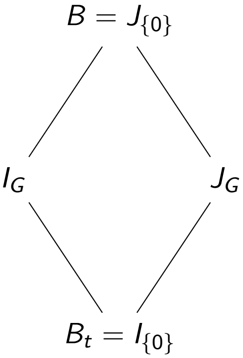
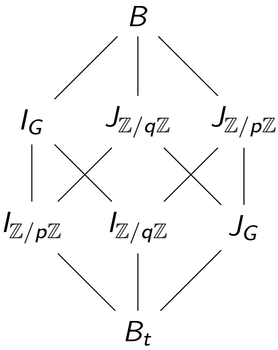


case $S = \mathbb{Z}/p\mathbb{Z}$



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The lattice of intermediate subfactors for $S = \mathbb{Z}/pqr\mathbb{Z}$

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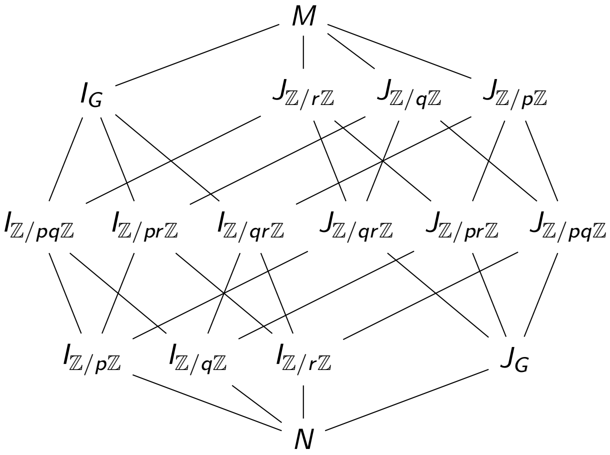

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